Statistics 330b/600b, Math 330b spring 2011
Homework # 6
Due: Thursday 24 February

Please attempt at least the starred problems.

*[1] Establish each of the following assertions. If you want to be efficient, you could try for a result that contains each of them as a special case.

(i) Let μ be a countably additive measure on $(\mathcal{X}, \mathcal{A})$. Suppose each singleton set $\{x\}$ belongs to $\mathcal{A}$. If μ is sigma-finite show that $\{x \in \mathcal{X} : \mu\{x\} > 0\}$ is at worst countably infinite. Hint: Reduce to the case where $\mu\mathcal{X} < \infty$ then consider the set $\{x \in \mathcal{X} : \mu\{x\} > 1/k\}$ for each k in $\mathbb{N}$.

(ii) Suppose $F : \mathbb{R} \to \mathbb{R}$ is a non-decreasing function. You may assume the following fact without proof: At each x, the right limit $F(x+) = \lim\{F(t) : t > x \text{ and } t \to x\}$ and the left limit $F(x-) = \lim\{F(t) : t < x \text{ and } t \to x\}$ both exist. Show that the set $\{x \in \mathbb{R} : F(x+) > F(x-)\}$ is at worst countably infinite.

(iii) Let P be a probability measure on the Borel sigma-field of a metric space $\mathcal{X}$. Write $B[x, r]$ for the ball $\{y \in \mathcal{X} : d(y, x) \le r\}$ and $\partial B[x, r]$ for its boundary. For each fixed x show that the set $\{r \in \mathbb{R}^+ : P\partial B[x, r] > 0\}$ is at worst countably infinite.

(iv) Let P be a probability measure on $\mathcal{B}(\mathbb{R}^2)$. Write $\mathcal{L}$ for the set of all straight lines in $\mathbb{R}^2$. Suppose P is non-atomic, that is, $P\{x\} = 0$ for all x in $\mathbb{R}^2$. Show that $\{L \in \mathcal{L} : PL > 0\}$ is at worst countably infinite.

*[2] Let $\mathcal{G}_k$ denote the set of all open subsets of $\mathbb{R}^k$, for $k = 1, 2$. Write $\mathcal{R}$ for set of "rational rectangles", that is, sets of the form $(r_1, r_2) \times (r_3, r_4)$ with each r_i rational. Prove that $\mathcal{B}(\mathbb{R}^2) = \mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R})$ by the following steps.

(i) For each G in $\mathcal{G}_2$, show that $G = \cup\{R \in \mathcal{R} : G \supseteq R\}$. Deduce that $\mathcal{B}(\mathbb{R}^2) \subseteq \mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R})$.

(ii) Let $\mathcal{A} = \{A \in \mathcal{B}(\mathbb{R}) : A \times \mathbb{R} \in \mathcal{B}(\mathbb{R}^2)\}$. Show that $\mathcal{A} = \mathcal{B}(\mathbb{R})$. Hint: Start by showing $\mathcal{A} \supset \mathcal{G}_1$.

(iii) From part (ii), and its analog for sets of the form $\mathbb{R} \times A$, deduce that $\mathcal{B}(\mathbb{R}^2) \supseteq \mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R})$.

[3] Suppose X is a real-valued random variable defined on $(\Omega, \mathcal{F}, \mathbb{P})$. The moment generating function for X is defined as $M(t) := \mathbb{P}e^{tX}$ for $t \in \mathbb{R}$.

(i) Show that the set $\mathbb{J} := \{t \in \mathbb{R} : M(t) < \infty\}$ is convex.

(ii) Suppose $\mathbb{J}$ has a non-empty interior, $\text{int}(\mathbb{J})$. Justify the interchange of differentiation and integration to show that M has derivatives $M'(t) = \mathbb{P}Xe^{tX}$ and $M''(t) = \mathbb{P}X^2e^{tX}$ on $\text{int}(\mathbb{J})$. Hint: I found it helpful to consider a t_0 and a $\delta > 0$ for which $[t_0 - 2\delta, t_0 + 2\delta] \subset \mathbb{J}$. For $|t - t_0| \le \delta$, bound $|X|^k e^{tX}$ by a constant times $e^{(t_0 - 2\delta)X} + e^{(t_0 + 2\delta)X}$.

(iii) Define $\psi(t) := \log M(t)$. For $t \in \text{int}(\mathbb{J})$, show that the second derivative of ψ equals the variance of X with respect to the probability measure $\mathbb{P}_t$ defined by $d\mathbb{P}_t/d\mathbb{P} = e^{tX}/M(t)$. Deduce that ψ is convex, at least on $\text{int}(\mathbb{J})$.