Statistics 330b/600b, Math 330b spring 2011
Homework # 7
Due: Thursday 3 March

Please attempt at least the starred problems.

*[1] Suppose $S_i = \xi_1 + \cdots + \xi_i$ for $i = 1, 2, \ldots, n$, where $\{\xi_j : j = 1, 2, \ldots, n\}$ are iid with $\mathbb{P}\{\xi_j = -1\} = 1/2 = \mathbb{P}\{\xi_j = +1\}$.

(i) For each ω define $\sigma(\omega)$ as the smallest j for which $S_j(\omega) = \max_i S_i(\omega)$. Explain why $S_\sigma \geq S_1$ everywhere, with strict inequality on a set with probability at least 1/2. Deduce that $\mathbb{P}S_\sigma > \mathbb{P}S_1$. Why does this result not contradict the Stopping Time Lemma?

(ii) Explain why the following argument is wrong. Define $\tau = \inf\{i : S_i \geq 2\}$. Clearly $S_\tau \geq 2$ almost surely, which implies $\mathbb{P}S_\tau \geq 2 > \mathbb{P}S_1$.

(iii) Suppose we actually have an infinite sequence of iid ξ_j's with $\mathbb{P}\{\xi_j = -1\} = 1/2 = \mathbb{P}\{\xi_j = +1\}$. Define $S_i = \xi_1 + \cdots + \xi_i$ for $i \in \mathbb{N}$. Define $\tau = \inf\{i : S_i \geq 2\}$. Clearly $S_\tau \geq 2$ almost surely, which implies $\mathbb{P}S_\tau \geq 2 > \mathbb{P}S_1$. Why does this result not contradict the Stopping Time Lemma?

*[2] Suppose $\mathcal{A}_i$ is a sigma-field on a set $\mathcal{X}_i$, for $i = 0, 1, 2$. Suppose also that $T_i : \mathcal{X}_0 \to \mathcal{X}_i$ is $\mathcal{A}_0\backslash\mathcal{A}_i$-measurable, for $i = 1, 2$. Define $S : \mathcal{X}_0 \to \mathcal{X}_1 \times \mathcal{X}_2$ by $S(x) = (T_1(x), T_2(x))$. Show that S is $\mathcal{A}_0\backslash\mathcal{A}_1 \otimes \mathcal{A}_2$-measurable.

[3] Suppose W and Z are nonnegative random variables with $\|Z\|_p < \infty$ for some $p > 1$. Suppose also that there exists positive constants β and C for which

$$t\mathbb{P}\{W > \beta t\} \leq C\mathbb{P}Z\{W > t\} \qquad \text{for all } t > 0.$$

Show that $\|W\|_p \leq Cp\beta^p \|Z\|_p /(p-1)$. *To make things slightly easier, I will let you assume that* $\|W\|_p < \infty$. *For a truly virtuoso effort, you might also show that the finiteness of* $\|W\|_p$ *actually follows from the finiteness of* $\|Z\|_p$.

*[4] (Doob's $\mathcal{L}^p$ maximal inequality) UGMTP Problem 6.9. You may appeal to the result from Problem 3 to eliminate some of the steps suggested in the book.