Statistics 330b/600b, Math 330b spring 2011
Homework # 8
Due: Thursday 24 March

Please attempt at least the starred problems.

*[1] Suppose $\{(X_n, \mathcal{F}_n) : n \in \mathbb{N}_0\}$ is a nonnegative supermartingale. Define stopping times $\sigma = \inf\{i : X_i = 0\}$ and $\tau_\epsilon = \inf\{i \geq \sigma : X_i \geq \epsilon\}$, for each fixed $\epsilon > 0$.

(i) Adapt the argument used for Dubins's inequality to show that $\mathbb{P}\{\tau_\epsilon \leq N\} = 0$ for each fixed $N \in \mathbb{N}$.

(ii) Deduce that $\tau_\epsilon = \infty$ almost surely, for each fixed $\epsilon > 0$.

(iii) For almost all ω, if $\sigma(\omega) < \infty$ deduce that $X_j(\omega) = 0$ for all $j \geq \sigma(\omega)$.

*[2] Suppose $\{(X_n, \mathcal{F}_n) : n \in \mathbb{N}_0\}$ is a martingale with $\sup_n \mathbb{P}X_n^2 < \infty$. For the following arguments, it helps to work with the martingale differences $\xi_i := X_i - X_{i-1}$ and their variances $v_i := \mathbb{P}\xi_i^2$.

(i) Show that $V_n := \mathbb{P}X_n^2 = \mathbb{P}X_0^2 + \sum_{1 \leq i \leq n} v_i$.

(ii) Deduce that $\{X_n : n \in \mathbb{N}_0\}$ is a Cauchy sequence in $\mathcal{L}^2$, implying existence of an X_∞ in $\mathcal{L}^2$ for which $\|X_n - X_\infty\|_2 \to 0$.

(iii) Show that $V_\infty := \mathbb{P}X_\infty^2 = \mathbb{P}X_0^2 + \sum_{i \in \mathbb{N}} v_i$.

(iv) For each m, n in $\mathbb{N}$ for which $n < m$ define $\Delta_{n,m} := \max_{n \leq i \leq m} |X_i - X_n|$ and $M_{n,m} := \max_{n \leq i \leq m} X_i$. Use Doob's inequality to show that

$$\mathbb{P}|M_{n,m} - X_n|^2 \leq \mathbb{P}\Delta_{n,m}^2 \leq 4(V_m - V_n)$$

(v) For each $\epsilon > 0$ deduce that

$$\epsilon^2 \mathbb{P}\{M_{n,m} > \epsilon + X_\infty\} \leq \mathbb{P}|M_{n,m} - X_\infty|^2 \leq 9(V_\infty - V_n)$$

(vi) By a careful passage to the limit (as $m \to \infty$, then $n \to \infty$, then $\epsilon \to 0$) deduce that $\limsup_i X_i \leq X_\infty$ almost surely.

(vii) Combine the result from part (vi) with an analogous result for the martingale $\{(-X_n, \mathcal{F}_n) : n \in \mathbb{N}_0\}$ to conclude that $X_n \to X_\infty$ almost surely.