Statistics 330b/600b, Math 330b spring 2011
Homework # 9
Due: Thursday 31 March

Please attempt at least the starred problems. Assume that all random variables are defined on the same Ω, for a given probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

[1] Suppose $X, X_1, X_2, \ldots$ are real valued random variables. In class I used Dominated Convergence to show that $X_n \to X$ in probability if $X_n \to X$ almost surely. Here are some similar facts.

(i) If $X_n \to X$ in probability show that there exists a subsequence $\{X_{n(i)} : i \in \mathbb{N}\}$ that converges to X almost surely. Hint: Start by explaining why there exists an $n(i)$ for which $\mathbb{P}\{|X_n - X| > 2^{-i}\} < 2^{-i}$ for all $n \geq n(i)$.

(ii) If there exists a $\delta > 0$ for which $\mathbb{P}\{|X_n - X| > \delta\} \geq \delta$ for all n, show that there exists no subsequence along which $X_{n(i)} \to X$ almost surely.

(iii) Show that $X_n \to X$ in probability if and only if for each subsequence $\{X_{n(i)} : i \in \mathbb{N}\}$ there exists a sub-subsequence $\{X_{n(i_j)} : j \in \mathbb{N}\}$ that convergences almost surely to X.

[2] Suppose $X_n \to X$ in probability. Use the results from Problem 1 to establish the following facts.

(i) If f is a measurable real valued function that is continuous at each point of some measurable set C for which $\mathbb{P}\{X \in C\} = 1$, show that $f(X_n) \to f(X)$ in probability.

(ii) Suppose there exists an integrable random variable W for which $W \geq |X_n|$ almost surely for each n. Show that $\mathbb{P}|X_n - X| \to 0$.

*[3] Suppose $\{A_n : n \in \mathbb{N}\}$ is a sequence of exchangeable events, that is, for each k, the joint distribution of $A_1, \ldots, A_k$ is the same as the joint distribution of $A_{i_1}, \ldots, A_{i_k}$ for all distinct $i_1, \ldots, i_k$. Define $T_n = (A_1 + \cdots + A_n)/n$.

(i) For $n < m$ show that $\mathbb{P}|T_n - T_m|^2 \leq n^{-1} + m^{-1}$. Deduce that there exists a random variable T for which $\mathbb{P}|T_n - T|^2 \to 0$.

Write Q for the distribution of T. You may assume that conditional distributions $\{\mathbb{P}_t : 0 \leq t \leq 1\}$ exist. That is, $\mathbb{P}F = Q^t \mathbb{P}_t F$ for each F in $\mathcal{F}$.

(ii) In class I argued that $\mathbb{P}_t(A_i) = t$ almost surely, as follows: *For each bounded continuous real function g on $[0, 1]$, I argued informally that*

$$\mathbb{P}A_i g(T_n) = \mathbb{P}T_n g(T_n) \to \mathbb{P}T g(T) = Q^t t g(t)$$
$$\mathbb{P}A_i g(T_n) \to \mathbb{P}A_i g(T) = Q^t (g(t) \mathbb{P}_t A_i)$$

I then asserted that, by sort sort of generating class argument, it follows that $\mathbb{P}_t A_1 = t$ a.e. $[Q]$. Make the whole argument from class completely rigorous.

(iii) Extend the argument to prove that $\mathbb{P}_t(A_i A_j A_k) = t^3$ a.e. $[Q]$ each triple (i, j, k) of distinct positive integers.