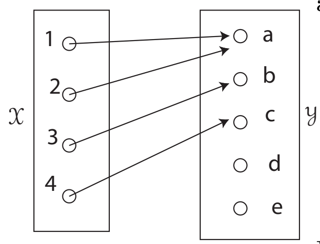
**Statistics 330b/600b, Math 330b spring 2013**
Homework # 1
Due: Thursday 24 January

*Please attempt at least the starred problems.*

*[1] Suppose $T$ maps a set $\mathcal{X}$ into a set $\mathcal{Y}$. For each $B \subseteq \mathcal{Y}$ and $A \subseteq \mathcal{X}$ define $T^{-1}B := \{x \in \mathcal{X} : T(x) \in B\}$. and $T(A) := \{T(x) : x \in A\}$. Some of the following eight assertions are true in general and some are false.




$$
\begin{array}{llll}
(i) & T\left(\cup_i A_i\right) = \cup_i T(A_i) & (ii) & T^{-1}\left(\cup_i B_i\right) = \cup_i T^{-1}(B_i) \\
(iii) & T\left(\cap_i A_i\right) = \cap_i T(A_i) & (iv) & T^{-1}\left(\cap_i B_i\right) = \cap_i T^{-1}(B_i) \\
(v) & T\left(A^c\right) = \left(T\left(A\right)\right)^c & (vi) & T^{-1}\left(B^c\right) = \left(T^{-1}\left(B\right)\right)^c \\
(vii) & T^{-1}\left(T(A)\right) = A & (viii) & T\left(T^{-1}(B)\right) = B
\end{array}
$$

Provide counterexamples for each of the false assertions. You might find it helpful to contemplate the picture for a special case.

*[2] A field on $\mathcal{X}$ is an $\mathcal{F} \subseteq 2^{\mathcal{X}}$ (that is, $\mathcal{F}$ is a set of subsets of $\mathcal{X}$) that contains $\emptyset$ and is stable under complements, finite unions, and finite intersections. If $\mathcal{E} \subseteq 2^{\mathcal{X}}$ then

$$\text{FIELD}(\mathcal{E}) := \bigcap \{\mathcal{D} : \mathcal{D} \supseteq \mathcal{E} \text{ and } \mathcal{D} \text{ is a field on } \mathcal{X}\}$$

is the smallest field $\mathcal{D}$ on $\mathcal{X}$ for which $\mathcal{D} \supseteq \mathcal{E}$. (Proof?)

In class, I outlined a proof of the following fact for each probability measure $\mu$ on $\sigma(\mathcal{E})$: for each $A \in \sigma(\mathcal{E})$ and each $\epsilon > 0$ there exists and $F$ in $\text{FIELD}(\mathcal{E})$ for which $\mu(A \Delta F) < \epsilon$. The proof started with: let $\mathcal{A}_0$ be the set of all $A$'s in $\sigma(\mathcal{E})$ with the desired property. Then I proceeded to argue that $\mathcal{E} \subseteq \mathcal{A}_0$ and $\mathcal{A}_0$ is a sigma-field. Along the way I skipped over a few points. Write out a complete proof with all the details. Please use only Boolean algebra and the properties of $\mu$ as a measure on $\sigma(\mathcal{E})$; don't use any of the integral properties that make the proof so much easier.

[3] Suppose a set $\mathcal{E}$ of subsets of $\mathcal{X}$ cannot separate a particular pair of points $x$, $y$, that is, for every $E$ in $\mathcal{E}$, either $\{x, y\} \subseteq E$ or $\{x, y\} \subseteq E^c$. Show that $\sigma(\mathcal{E})$ also cannot separate the pair. Hint: Consider $\mathcal{D} := \{A \in \sigma(\mathcal{E}) : A \text{ does not separate } x \text{ and } y\}$.

[4] For the extended real line $\overline{\mathbb{R}} = [-\infty, \infty]$, define $\mathcal{E} := \{[-\infty, t] : t \in \overline{\mathbb{R}}\}$. Show that a subset $A$ of $\overline{\mathbb{R}}$ belongs to $\mathcal{A} := \sigma(\mathcal{E})$ if and only if $A \cap \mathbb{R} \in \mathcal{B}(\mathbb{R})$.