Statistics 330b/600b, Math 330b spring 2013
Homework # 10
Due: Thursday 18 April

Please attempt at least the starred problems.

*[1] For a fixed n let μ denote counting measure on $\mathcal{X} = \{0, 1, \dots, n\}$. For each θ in $\Theta = (0, 1)$ let $\mathbb{P}_\theta$ denote the $\text{Bin}(n, \theta)$ probability measure, which has density

$$p(x, \theta) = \binom{n}{x} \theta^x (1 - \theta)^{n-x} \qquad \text{with respect to } \mu.$$

Let T denote the map that projects $\mathcal{X} \times \Theta$ onto $\mathcal{X}$. That is, $T(x, \theta) = x$. Suppose the prior π has density

$$g(\theta) = \theta^{-1}(1 - \theta)^{-1} \qquad \text{with respect to Lebesgue measure } \mathfrak{m} \text{ on } \mathcal{B}(\Theta)$$

Let $\lambda = \mu \otimes \mathfrak{m}$ and $\mathbb{P} = \pi \otimes \mathcal{P}$ where $\mathcal{P} = \{\mathbb{P}_\theta : \theta \in \Theta\}$. *[Despite the choice of letter, $\mathbb{P}$ is not a probability measure.]*

(i) Find the density of $\mathbb{P}$ with respect to $\mu \otimes \mathfrak{m}$.

(ii) Define $\mathbb{Q} = T\mathbb{P}$. Find $\mathbb{Q}\{x\}$ for each x in $\mathcal{X}$.

(iii) Find a collection of measures $\Gamma = \{\gamma_x : x \in \mathcal{X}\}$ on $\mathcal{B}(\Theta)$ for which $\mathbb{P} = \mu \otimes \Gamma$. That is, $\mathbb{P} f(x, \theta) = \mu^x \gamma_x^\theta f(x, \theta)$ for each nonnegative, jointly measurable f on $\mathcal{X} \times \Theta$. *[In other words, the set of measures $\{\delta_x \otimes \gamma_x : x \in \mathcal{X}\}$ is a T, μ-disintegration of $\mathbb{P}$.]* You should be able to spot a suitable Γ without much calculation.

(iv) Show that there does not exist a $T, \mathbb{Q}$-disintegration of $\mathbb{P}$. Hint: Consider what such a disintegration would say about $\mathbb{P}\{x = 0, |\theta| < 1/2\}$.

> **Remark.** The measure γ_x plays the role of the posterior distribution. For example, in decision theory using squared error loss, the function $\delta(x)$ that minimizes $\gamma_x^\theta(\theta - \delta(x))^2$ for each x has useful properties.

*[2] Let μ denote counting measure on $\{1, 2\}$ and λ denote Lebesgue measure on $\mathcal{B}(\mathbb{R})$. For each $\theta = (\alpha, \gamma_1, \gamma_2)$ in $\Theta := (0, 1) \times \mathbb{R}^2$ define

$$f(x, y, \theta) = \alpha\{y = 1\}\phi(x - \gamma_1) + (1 - \alpha)\{y = 2\}\phi(x - \gamma_2),$$

a probability density with respect to $\lambda \times \mu$. (Here ϕ is the $N(0, 1)$ density function.) The marginal density for x is the mixture $\alpha\phi(x - \gamma_1) + (1 - \alpha)\phi(x - \gamma_2)$.

For $\mathbf{x} = (x_1, \dots, x_n)$ and $\mathbf{y} = (y_1, \dots, y_n)$ define

$$p(\mathbf{x}, \mathbf{y}, \theta) = \prod_{i \le n} f(x_i, y_i, \theta)$$

(i) Find the maximium likelihood estimator for θ based on $\mathbf{x}$ and $\mathbf{y}$.

(ii) Now suppose we only observe $\mathbf{x}$. The marginal density for $\mathbf{x}$ is $\nu^{\mathbf{y}} p(\mathbf{x}, \mathbf{y}, \theta)$ where ν is the n-fold product measure μ^n. The EM algorithm iteratively updates an estimate θ_0 by the value θ_1 that maximimizes the function

$$H_{\theta_0}(\theta) = \mathbb{P}_{\theta_0}\left(\log p(\mathbf{x}, \mathbf{y}, \theta) \mid \mathbf{x}\right).$$

Find the value for θ_1.