Statistics 330b/600b, Math 330b spring 2013
Homework # 2
Due: Thursday 31 January

Please attempt at least the starred problems.

*[1] Suppose $\mathcal{A}$ is a sigma-field on a set $\mathcal{X}$ and μ is a measure on $\mathcal{A}$. Write $\mathcal{N}$ for $\{N \in \mathcal{A} : \mu N = 0\}$. Define

$$\mathcal{A}_\mu := \{B \subseteq \mathcal{X} : \exists A \in \mathcal{A},\, N \in \mathcal{N} \text{ such that } |B - A| \leq N\}.$$

(i) Show that $\mathcal{A}_\mu$ is a sigma-field. Hint: If $\{B_i : i \in \mathbb{N}\} \subseteq \mathcal{A}_\mu$ and $|B_i - A_i| \leq N_i$, show that $|\cup_i B_i - \cup_i A_i| \leq \cup_i N_i$.

(ii) If $|B - A_i| \leq N_i$ for $i = 1, 2$, with $A_i \in \mathcal{A}$ and $N_i \in \mathcal{N}$, show that $\mu A_1 = \mu A_2$. Hint: Show $A_1 \leq A_2 + N_1 + N_2$.

(iii) If $|B - A| \leq N$ with $A \in \mathcal{A}$ and $N \in \mathcal{N}$ define $\nu B = \mu A$. Show that ν is a measure on $\mathcal{A}_\mu$ whose restriction to $\mathcal{A}$ equals μ.

*[2] Suppose $f_1, \ldots, f_k \in \mathcal{M}^+(\mathcal{X}, \mathcal{A})$ and $\theta_1, \ldots, \theta_k$ are strictly positive numbers that sum to one. Let μ be a measure on $\mathcal{A}$. Show that

$$\mu \prod\nolimits_{i \leq k} f_i^{\theta_i} \leq \prod\nolimits_{i \leq k} (\mu f_i)^{\theta_i}$$

by following these steps.

(i) Explain why the inequality is trivially true if $\mu f_i = 0$ for at least one i or if $\mu f_i = \infty$ for at least one i (and all the other μf_j are strictly positive).

(ii) Explain why there is no loss of generality in assuming that $\mu f_i = 1$ for each i and $f_i(x) < \infty$ for each x and i.

(iii) For all $a_1, \ldots, a_k \in \mathbb{R}^+$, show that $\prod_{i \leq k} a_i^{\theta_i} \leq \sum_{i \leq k} \theta_i a_i$. Hint: First dispose of the trivial case where at least one a_i is zero, then rewrite the inequality using $b_i = \log a_i$. *You do not need to reprove that the* log *function is concave on* $(0, \infty)$.

(iv) Complete the proof by considering the inequality from (iii) with $a_i = f_i(x)$.

> **Remark.** Textbooks often contain the the special case where $k = 2$ and $\theta_1 = 1/p$ and $\theta_2 = 1/q$ and $f_1 = |g_1|^p$ and $f_2 = |g_2|^q$, with the assertion that $|\mu(g_1 g_2)| \leq \mu|g_1 g_2| \leq (\mu|g_1|^p)^{1/p} (\mu|g_2|^q)^{1/q}$.

[3] Suppose $f_1, \ldots, f_n$ are functions in $\mathcal{M}^+$ and μ is a measure for which $\mu f_i < \infty$ for each i. Show that

$$\begin{aligned} &\sum\nolimits_i \mu f_i - \sum\nolimits_{i<j} \mu(f_i \wedge f_j) \\ &\quad \leq \mu\left(\max\nolimits_i f_i\right) \\ &\quad \leq \sum\nolimits_i \mu f_i - \sum\nolimits_{i<j} \mu(f_i \wedge f_j) + \sum\nolimits_{i<j<k} \mu(f_i \wedge f_j \wedge f_k). \end{aligned}$$

Hint: For fixed nonnegative numbers $a_1, \ldots, a_n$ establish inequalities like $\sum_i a_i \leq \max_i a_i + \sum_{i<j} a_i \wedge a_j$. For simplicity assume $a_1 \geq a_2 \geq \cdots \geq a_n \geq 0$ then consider the coefficients of each a_k.