Statistics 330b/600b, Math 330b spring 2013
Homework # 3
Due: Thursday 7 February

Please attempt at least the starred problems.

*[1] For f in $\mathcal{L}^1(\mu)$ define $\|f\|_1 = \mu|f|$. Let $\{f_n\}$ be a Cauchy sequence in $\mathcal{L}^1(\mu)$, that is, $\|f_n - f_m\|_1 \to 0$ as $\min(m,n) \to \infty$. Show that there exists an f in $\mathcal{L}^1(\mu)$ for which $\|f_n - f\|_1 \to 0$, by following these steps. Note: Don't confuse Cauchy sequences (in $\mathcal{L}^1$ distance) of functions with Cauchy sequences of real numbers.

(i) Find an increasing sequence $\{n(k)\}$ such that $\sum_{k=1}^{\infty} \left\|f_{n(k)} - f_{n(k+1)}\right\|_1 < \infty$. Deduce that the function $H := \sum_{k=1}^{\infty} |f_{n(k)} - f_{n(k+1)}|$ is integrable.

(ii) Show that there exists a real-valued, measurable function f for which

$$H \geq |f_{n(k)}(x) - f(x)| \to 0 \qquad \text{as } k \to \infty, \text{ for } \mu \text{ almost all } x.$$

Deduce that $\left\|f_{n(k)} - f\right\|_1 \to 0$ as $k \to \infty$.

(iii) Show that f belongs to $\mathcal{L}^1(\mu)$ and $\|f_n - f\|_1 \to 0$ as $n \to \infty$.

*[2] Let Ψ be a convex, increasing function for which $\Psi(0) = 0$ and $\Psi(x) \to \infty$ as $x \to \infty$. (For example, $\Psi(x)$ could equal x^p for some fixed $p \geq 1$, or $\exp(x) - 1$ or $\exp(x^2) - 1$.) Define $\mathcal{L}^{\Psi}(\mathfrak{X}, \mathcal{A}, \mu)$ to be the set of all real-valued measurable functions on $\mathfrak{X}$ for which $\mu\Psi(|f|/c_0) < \infty$ for at least one positive real constant c_0. Define $\|f\|_{\Psi} := \inf\{c > 0 : \mu\Psi(|f|/c) \leq 1\}$, with the convention that the infimum of an empty set equals $+\infty$. For each f, g in $\mathcal{L}^{\Psi}(\mathfrak{X}, \mathcal{A}, \mu)$ and each real t prove the following assertions.

(i) $\|f\|_{\Psi} < \infty$. Hint: Apply Dominated Convergence to $\mu\Psi(|f|/c)$.

(ii) $f + g \in \mathcal{L}^{\Psi}(\mathfrak{X}, \mathcal{A}, \mu)$ and the triangle inequality holds: $\|f + g\|_{\Psi} \leq \|f\|_{\Psi} + \|g\|_{\Psi}$. Hint: If $c > \|f\|_{\Psi}$ and $d > \|g\|_{\Psi}$, deduce that

$$\Psi\left(\frac{|f+g|}{c+d}\right) \leq \frac{c}{c+d}\Psi\left(\frac{|f|}{c}\right) + \frac{d}{c+d}\Psi\left(\frac{|g|}{d}\right),$$

by convexity of Ψ.

(iii) $tf \in \mathcal{L}^{\Psi}(\mathfrak{X}, \mathcal{A}, \mu)$ and $\|tf\|_{\Psi} = |t| \; \|f\|_{\Psi}$ for each $t \in \mathbb{R}$.

> **Remark.** $\|\cdot\|_{\Psi}$ is called an Orlicz "norm"—to make it a true norm one should work with equivalence classes of functions equal μ almost everywhere. The $\mathcal{L}^p$ norms correspond to the special case $\Psi(x) = x^p$, for some $p \geq 1$.

[3] Suppose F is a nonnegative function in $\mathcal{L}^1(\mathfrak{X}, \mathcal{A}, \mu)$.

(i) For each $\epsilon > 0$ show that there exists a $\delta > 0$ such that $\mu(FA) < \epsilon$ for every A in $\mathcal{A}$ for which $\mu A < \delta$. Hint: If $\{A_n\}$ is a sequence in $\mathcal{A}$ for which $\mu A_n < 2^{-n}$ and $\mu(FA_n) \geq \epsilon$, what do you know about the sequence $B_n := \cup_{i \geq n} A_i$?

(ii) Suppose $\{A_n\}$ is a sequence in $\mathcal{A}$ for which $\mu\left(A_n\{F > \gamma\}\right) \to 0$ for each constant $\gamma > 0$. Show that $\mu\left(FA_n\right) \to 0$.