Statistics 330b/600b, Math 330b spring 2013
Homework # 4
Due: Thursday 14 February

Please attempt at least the starred problems.

*[1] Let $D = \{(x, y) \in \mathbb{R}^2 : x = y\}$.

(i) For each pair A, B in $\mathcal{B}(\mathbb{R})$, show that $D \neq A \times B$.

(ii) Show that $D \in \mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R})$.

[2] Suppose $\mathcal{E}_0$ is a (nonempty) collection of subsets of some set $\mathcal{X}$. For each $n \in \mathbb{N}$ define $\mathcal{E}_n$ to be the collection of all subsets of the form E_1^c or $E_1 \cup E_2$ or $E_1 \cap E_2$ with $E_i \in \mathcal{E}_{n-1}$. Define $\mathcal{F} := \bigcup_{n \in \mathbb{N}} \mathcal{E}_n$.

(i) Show that $\mathcal{F}$ is a field on $\mathcal{X}$. (That is, $\emptyset \in \mathcal{F}$ and $\mathcal{F}$ is stable under finite unions, finite intersections, and complements.)

(ii) If $\mathcal{G}$ is a field on $\mathcal{X}$ for which $\mathcal{G} \supseteq \mathcal{E}_0$, show that $\mathcal{G} \supseteq \mathcal{F}$.

(iii) If $\mathcal{E}_0$ is countable show that $\mathcal{F}$ is countable.

(iv) A sigma field $\mathcal{A}$ on $\mathcal{X}$ is said to be ***countably generated*** if $\mathcal{A} = \sigma(\mathcal{E})$ for some countable $\mathcal{E}$. Explain why $\mathcal{E}$ can always be assumed to be a field.

*[3] Suppose $\mathcal{A}$ is a sigma-field on a set $\mathcal{X}$ and $\mathcal{B}$ is a countably generated sigma-field on a set $\mathcal{Y}$. Suppose also that $\mathcal{B}$ separates the points of $\mathcal{Y}$, in the sense of HW1.2: if $y_1 \neq y_2$ then there exists a set $B \in \mathcal{B}$ for which $y_1 \in B$ and $y_2 \in B^c$. For each $\mathcal{A}\backslash\mathcal{B}$-measurable map T from $\mathcal{X}$ to $\mathcal{Y}$ show that the set $D = \{(x, y) \in \mathcal{X} \times \mathcal{Y} : y = Tx\}$ belongs to $\mathcal{A} \otimes \mathcal{B}$. Hint: If $\mathcal{B}$ is generated by a countable field $\mathcal{F}$, consider the set $\bigcup_{F \in \mathcal{F}} (T^{-1}F^c) \times F$.

[4] Suppose $\mathcal{A}$ and $\mathcal{B}$ are countably generated sigma-fields, on the sets $\mathcal{X}$ and $\mathcal{Y}$ respectively. Show that $\mathcal{A} \otimes \mathcal{B}$ is countably generated.

[5] Let $\mathcal{X}$ and $\mathcal{Y}$ be topological spaces equipped with their Borel sigma-fields $\mathcal{B}(\mathcal{X})$ and $\mathcal{B}(\mathcal{Y})$. Equip $\mathcal{X} \times \mathcal{Y}$ with the product topology and its Borel sigma-field $\mathcal{B}(\mathcal{X} \times \mathcal{Y})$. (The open sets in the product space are, by definition, all possible unions of sets $G \times H$, with G open in $\mathcal{X}$ and H open in $\mathcal{Y}$.)

(i) Show that $\mathcal{B}(\mathcal{X}) \otimes \mathcal{B}(\mathcal{Y}) \subseteq \mathcal{B}(\mathcal{X} \times \mathcal{Y})$.

(ii) A topology $\mathcal{G}_1$ is said to be countably generated if there exists a countable $\mathcal{G}_2 \subseteq \mathcal{G}_1$ such that $G = \cup\{H \in \mathcal{G}_2 : H \subseteq G\}$ for each $G \in \mathcal{G}_1$. If both $\mathcal{X}$ and $\mathcal{Y}$ have countably generated topologies, prove equality of the two sigma-fields (from part (ii)) on the product space.

(iii) Show that $\mathcal{B}(\mathbb{R}^n) = \mathcal{B}(\mathbb{R}^k) \otimes \mathcal{B}(\mathbb{R}^{n-k})$.

*[6] Suppose X is a real valued random variable, defined on a set Ω equipped with a sigma-field $\mathcal{F}$. Show that the set $\{(\omega, t) \in \Omega \times \mathbb{R} : X(\omega) > t\}$ belongs to $\mathcal{F} \otimes \mathcal{B}(\mathbb{R})$.

[7] Suppose μ and ν are finite measures on $\mathcal{B}(\mathbb{R})$. Let f be the indicator function of the set $\{(x, y) \in \mathbb{R}^2 : x = y\}$.

(i) Show that the set $A_\nu := \{x \in \mathbb{R} : \nu\{x\} > 0\}$ contains at most countably many points.

(ii) Show that $\mu \otimes \nu f(x, y) = \sum_{x \in A_\nu} \mu\{x\}\nu\{x\}$. Hint: First write $\nu^y f(x, y)$ as a sum of at most countably many terms.