Statistics 330b/600b, Math 330b spring 2013
Homework # 5
Due: Thursday 21 February

Please attempt at least the starred problems.

*[1] For each convex, real valued function Ψ on the real line there exists a countable family of linear functions for which $\Psi(x) = \sup_{i\in\mathbb{N}}(a_i + b_i x)$ for all x (see Appendix C of UGMTP). Use this representation to prove ***Jensen's inequality***: if $X \in \mathcal{L}^1(\Omega, \mathcal{F}, \mathbb{P})$, with $\mathbb{P}$ a probability measure, then $\mathbb{P}\Psi(X) \geq \Psi(\mathbb{P}X)$. You should first show that $\mathbb{P}\Psi(X)^- < \infty$, to ensure that $\mathbb{P}\Psi(X)$ is well defined.

[2] In class I outlined a proof (using the π–λ theorem) of the following result:

Let $\mathcal{E}_1, \dots, \mathcal{E}_n$ be classes of measurable sets, each class stable under finite intersections and containing the whole space Ω. If

$$\mathbb{P}(E_1 E_2 \dots E_n) = (\mathbb{P}E_1)(\mathbb{P}E_2)\dots(\mathbb{P}E_n) \qquad \text{for all } E_i \in \mathcal{E}_i, \text{ for } i = 1, 2, \dots, n,$$

then the sigma-fields $\sigma(\mathcal{E}_1), \sigma(\mathcal{E}_2), \dots, \sigma(\mathcal{E}_n)$ are independent.

Show that the stability under finite intersections is needed: try $\mathbb{P}$ as the uniform distribution on $\Omega = \{1,2,3,4\}$ and $\mathcal{E}_1 = \{\Omega, \{1,2\}\}$ and $\mathcal{E}_2 = \{\Omega, \{2,3\}, \{2,4\}\}$.

*[3] Let $A_1, A_2, \dots$ be events in a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Define $X_n = A_1 + \dots + A_n$ and $\sigma_n = \mathbb{P}X_n$.

(i) Show that $\|X_n/\sigma_n\|_2 \geq 1$. Hint: Jensen.

> **Remark.** If Y is a real-valued random variable for which $\mathbb{P}|Y|^2 < \infty$, its $\mathcal{L}^2$ norm is defined as $\|Y\|_2 := \left(\mathbb{P}|Y|^2\right)^{1/2}$. Compare with HW3.2 for $\Psi(x) = x^2$ or UGMTP Problem 2.17.)

(ii) Show that (as a pointwise inequality between random variables)

$$\{X_n = 0\} \leq \frac{(k - X_n)(k + 1 - X_n)}{k(k+1)}$$

for each positive integer k. Hint: Are there any values of X_n for which the ratio on the right-hand side is negative?

Now suppose $\sigma_n \to \infty$ and $\|X_n/\sigma_n\|_2 \to 1$.

(iii) By making an appropriate choice of the integer k (depending on n) in (ii), show that $\mathbb{P}\{X_n = 0\} \to 0$ as $n \to \infty$. Deduce that $\sum_1^\infty A_i \geq 1$ almost surely.

(iv) Prove that $\sum_{i=m}^\infty A_i \geq 1$ almost surely, for each fixed m. Hint: Show that the two convergence assumptions also hold for the sequence $A_m, A_{m+1}, \dots$.

(v) Deduce that $\mathbb{P}\{\omega \in A_i \text{ i. o. }\} = 1$.

(vi) If $\{B_i\}$ is a sequence of independent events for which $\sum_i \mathbb{P}B_i = \infty$, show that $\mathbb{P}\{\omega \in B_i \text{ i. o. }\} = 1$. *Please use (v). I am not interested in seeing the standard textbook proof for the harder direction of Borel-Cantelli.*

PTO

[4] Let $(\mathcal{X}, \mathcal{A}, \mu)$ and $(\mathcal{Y}, \mathcal{B}, \nu)$ be two measure spaces, with both μ and ν sigma-finite. Write $\mathcal{G}$ for the set of all functions expressible as finite linear combinations of measurable rectangles. That is, a typical g in $\mathcal{G}$ is expressible as a finite sum $\sum_{i=1}^{k} \alpha_i \{x \in A_i,\ y \in B_i\}$ for some sets $A_i \in \mathcal{A}$ and $B_i \in \mathcal{B}$ and real numbers α_i, for $i = 1, 2, \ldots, k$.

Show that for each f in $\mathcal{L}^1(\mathcal{X} \times \mathcal{Y}, \mathcal{A} \otimes \mathcal{B}, \mu \otimes \nu)$ and each $\epsilon > 0$ there exist a $g \in \mathcal{G}$ such that $\mu \otimes \nu |f - g| < \epsilon$. Follow these steps.

(i) First suppose that both μ and ν are finite measures and $|f|$ is bounded. Use a lambda-space argument to establish the asserted approximation property.

(ii) Extend to the sigma-finite case. Hint: First approximate the function f by some $f_n := (-n) \vee (f \wedge n)$.