Statistics 330b/600b, Math 330b spring 2013
Homework # 6
Due: Thursday 28 February

Please attempt at least the starred problems.

*[1] Suppose T is a function from a set $\mathcal{X}$ into a set $\mathcal{Y}$, which is equipped with a σ-field $\mathcal{B}$. Recall that $\sigma(T) := \{T^{-1}B : B \in \mathcal{B}\}$ is the smallest sigma-field on $\mathcal{X}$ for which T is $\sigma(T)\backslash\mathcal{B}$-measurable.

Show that to each f in $\mathcal{M}^+(\mathcal{X}, \sigma(T))$ there exists a g in $\mathcal{M}^+(\mathcal{Y}, \mathcal{B})$ such that $f = g \circ T$ (that is, $f(x) = g(T(x))$, for all x in $\mathcal{X}$) by following these steps.

(i) If f is the indicator function of $T^{-1}(B)$ and g is the indicator function of B, show that $f = g \circ T$.

(ii) Consider the case where $f \in \mathcal{M}^+_{\text{simple}}(\mathcal{X}, \sigma(T))$.

(iii) Suppose $f_n = g_n \circ T$ is a sequence in $\mathcal{M}^+_{\text{simple}}(\mathcal{X}, \sigma(T))$ that increases pointwise to f. Define $g(y) = \limsup g_n(y)$ for each y in $\mathcal{Y}$. Show that g has the desired property.

(iv) In part (iii), why can't we assume that $\lim g_n(y)$ exists for each y in $\mathcal{Y}$?

*[2] Suppose X is a random variable taking values in $[0, \infty)$ for which $\mathbb{P}X = \mu < \infty$. Let $X_1, X_2, \ldots$ be independent random variables, each with the same distribution as X. Define $Y_i := X_i\{X_i \le i\}$ and $\mu_i := \mathbb{P}Y_i$. Let $S_n := \sum_{i \le n} X_i$ and $T_n := \sum_{i \le n} Y_i$.

For a fixed $\rho > 1$, let $\{k_n\}$ be an increasing sequence of positive integers such that $k_n/\rho^n \to 1$.

(i) Show that there exists a finite constant C for which $\sum_{j \in \mathbb{N}} \{i \le k_j\}/k_j^2 \le C/i^2$ for each positive integer i and $\sum_{i \ge \ell} i^{-2} \le C/\ell$ for each positive integer ℓ.

(ii) Show that $\sum_{i \le n} \{X > i\} \le n \wedge X$. Deduce that

$$0 \le \mu - \mathbb{P}T_n/n = \mathbb{P}\sum_{i \le n} X\{X > i\}/n \to 0 \qquad \text{as } n \to \infty.$$

(iii) Show that $\sum_{i \in \mathbb{N}} \mathbb{P}\{X_i \ne Y_i\} \le \sum_{i \in \mathbb{N}} \mathbb{P}\{X > i\} < \infty$. Deduce that $(S_n - T_n)/n \to 0$ almost surely. Hint: $n^{-1}\sum_{i \le I} |X_i(\omega) - Y_i(\omega)| \to 0$ as $n \to \infty$, for each fixed I.

(iv) Show that $\text{var}(T_n) \le \sum_{i \le n} \mathbb{P}X^2\{X \le i\}$.

(v) Use parts (i) and (iv) to show that

$$\begin{aligned}\sum_{j \in \mathbb{N}} \mathbb{P}\{|T_{k_j} - \mathbb{P}T_{k_j}| > \epsilon k_j\} &\le \sum_{j \in \mathbb{N}} \sum_{i \in \mathbb{N}} \frac{\{i \le k_j\}\mathbb{P}X^2\{X \le i\}}{\epsilon^2 k_j^2} \\ &\le C\epsilon^{-2} \sum_{i \in \mathbb{N}} \mathbb{P}X^2\{X \le i\}/i^2 < \infty.\end{aligned}$$

Deduce that $(T_{k_j} - \mathbb{P}T_{k_j})/k_j \to 0$ almost surely as $j \to \infty$.

(vi) Deduce that $S_{k_j}/k_j \to \mu$ almost surely as $j \to \infty$.

(vii) For each $\rho' > \rho$, show that

$$\frac{S_{k_n}}{\rho' k_n} \le \frac{S_m}{m} \le \rho' \frac{S_{k_{n+1}}}{k_{n+1}} \qquad \text{for } k_n \le m \le k_{n+1},$$

when n is large enough.

(viii) Deduce that $\limsup S_m/m$ and $\liminf S_m/m$ both lie between μ/ρ' and $\mu\rho'$, with probability one.

(ix) Cast out a sequence of negligible sets as ρ decreases to 1 to deduce that $S_m/m \to \mu$ almost surely.

(x) Why does the SLLN for i.i.d. integrable random variables follow from the preceding argument?