Statistics 330b/600b, Math 330b spring 2013
Homework # 7
Due: Thursday 7 March

Please attempt at least the starred problems.

*[1] Suppose W and Z are nonnegative random variables with $\|Z\|_p < \infty$ for some $p > 1$. Suppose also that there exists positive constants β and C for which

$$t\mathbb{P}\{W > \beta t\} \le C\mathbb{P}Z\{W > t\} \qquad \text{for all } t > 0.$$

Show that $\|W\|_p \le Cp\beta^p \|Z\|_p /(p-1)$. *To make things slightly easier, I will let you assume that $\|W\|_p < \infty$. For a truly virtuoso effort, you might also show that the finiteness of $\|W\|_p$ actually follows from the finiteness of $\|Z\|_p$.*

[2] Suppose $\{(S_i, \mathcal{F}_i) : i = 1, 2, \dots, N\}$ is a nonnegative submartingale. For some fixed $p > 1$ suppose $\mathbb{P}S_i^p < \infty$ for each i. Let $q > 1$ be defined by $p^{-1} + q^{-1} = 1$. Show that $\mathbb{P}\left(\max_{i \le n} S_i^p\right) \le q^p \mathbb{P}S_n^p$, by following these steps.

(i) Write M_n for $\max_{i \le n} S_i$. For each fixed $x > 0$ show that

$$x\mathbb{P}\{M_n \ge x\} \le \mathbb{P}S_n\{M_n \ge x\}.$$

(ii) Invoke Problem [1].

*[3] Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. For each $n \in \mathbb{N}$ suppose $\mathcal{E}_n$ consists of finitely many disjoints sets from $\mathcal{F}$ that provide a partition of Ω, such that each E in $\mathcal{E}_n$ can be written as a disjoint union of sets from $\mathcal{E}_{n+1}$. (The partitions are nested.) For each $A \in \mathcal{F}$ and each $Y \in \mathcal{L}^1 = \mathcal{L}^1(\Omega, \mathcal{F}, \mathbb{P})$ define

$$\mathbb{P}_A Y = \begin{cases} \mathbb{P}(YA)/\mathbb{P}A & \text{if } \mathbb{P}A > 0 \\ 0 & \text{otherwise} \end{cases}$$

Define $\mathcal{F}_n := \sigma(\mathcal{E}_n)$. Define maps τ_n from $\mathcal{L}^1(\Omega, \mathcal{F}, \mathbb{P})$ into $\mathcal{L}^1(\Omega, \mathcal{F}_n, \mathbb{P}\big|_{\mathcal{F}_n})$ by

$$(\tau_n Y)(\omega) = \sum\nolimits_{E \in \mathcal{E}_n} \{\omega \in E\} \mathbb{P}_E Y.$$

Now consider a fixed nonnegative X in $\mathcal{L}^1$. Define $X_n = \tau_n(X)$.

(i) Show that $\{(X_n, \mathcal{F}_n) : n \in \mathbb{N}\}$ is a nonnegative martingale, which converges almost surely to some nonnegative, integrable random variable X_∞. Hint: Why is it enough to check that $\mathbb{P}X_n E = \mathbb{P}X_{n+1}E$ for each $E \in \mathcal{E}_n$?

(ii) Use Fatou's Lemma (UGMTP page 31) to deduce that $\mathbb{P}X \ge \mathbb{P}X_\infty$.

(iii) For each finite, nonnegative constant M show that there exists a nonnegative random variable $X_{\infty,M}$ for which $\mathbb{P}|\tau_n(X \wedge M) - X_{\infty,M}| \to 0$ as $n \to \infty$.

(iv) Show that $X_\infty \ge X_{\infty,M}$ almost surely for each M. Deduce that $\mathbb{P}X_\infty = \mathbb{P}X$.

(v) Show that $\mathbb{P}|X_n - X_\infty| \to 0$. Hint: Show that $\mathbb{P}(X_\infty - X_n)^+ \to 0$ as $n \to \infty$ and $\mathbb{P}(X_\infty - X_n)^+ = \mathbb{P}(X_\infty - X_n)^-$.

(vi) Define $\mathcal{E} = \cup_{n \in \mathbb{N}} \mathcal{E}_n$ and $\mathcal{F}_\infty = \sigma(\mathcal{E})$. Show that X_∞ is $\mathcal{F}_\infty$-measurable and $\mathbb{P}XE = \mathbb{P}X_\infty E$ for all $E \in \mathcal{E}$. Deduce that $\mathbb{P}X_\infty F = \mathbb{P}XF$ for all $F \in \mathcal{F}_\infty$.

The random variable X_n is usually called the conditional expectation of the random variable X given the sub-sigma-field $\mathcal{F}_n$ and X_∞ is called its conditional expectation given $\mathcal{F}_\infty$.