Statistics 330b/600b, Math 330b spring 2013
Homework # 8
Due: Thursday 4 April

Please attempt at least the starred problems.

*[1] Suppose ν and μ are finite measures defined on a sigma-field $\mathcal{A}$ of subsets of a set $\mathcal{X}$. In class I showed that, if $\nu f \le \mu f$ for each $f \in \mathcal{M}^+(\mathcal{X},\mathcal{A})$ then there exists an $\mathcal{A}$-measurable function Δ for which $0 \le \Delta \le 1$ and $\nu f = \mu(f\Delta)$ for all f in $\mathcal{M}^+(\mathcal{X},\mathcal{A})$.

(i) Extend this result to the case where ν and μ are still finite measures but $\nu \ll \mu$. Hint: Define $\lambda = \nu + \mu$ then invoke the result from class to show that there is an $\mathcal{A}$-measurable function Δ_0 for which $0 \le \Delta_0 \le 1$ and $\nu f = \mu(f\Delta_0)$ for all f in $\mathcal{M}^+(\mathcal{X},\mathcal{A})$. Then show that $\mu\Delta_0\{\Delta_0 \ge 1\} = 0$. Finally explain why $\nu f = \mu(f\Delta)$ for all $f \in \mathcal{M}^+(\mathcal{X},\mathcal{A})$, where

$$\Delta := \frac{\Delta_0}{1-\Delta_0}\{\Delta_0 < 1\}$$

Beware of $\infty - \infty$.

(ii) Show that the Δ from part (ii) is unique up to μ-equivalence.

(iii) Extend the theorem to the case where $\nu = \sum_{i\in\mathbb{N}} \nu_i$, a sum of finite measures for which $\nu_i \ll \mu$ for each i.

[2] Suppose ν and μ are measures on $(\mathcal{X},\mathcal{A})$ for which $\nu A = 0$ for every $A \in \mathcal{A}$ such that $\mu A = 0$. Show that $\nu f = 0$ for every $f \in \mathcal{M}^+(\mathcal{X},\mathcal{A})$ for which $\mu f = 0$.

*[3] Let $\mathcal{K}$ be a closed, convex subset of $\mathcal{H} := \mathcal{L}^2(\mathcal{X},\mathcal{A},\mu)$. Show that to each f in $\mathcal{H}$ there is an f_0 (unique up to μ-equivalence) in $\mathcal{K}$ for which

<1> $$\|f - f_0\|_2 = \delta := \inf\{\|f-h\|_2 : h \in \mathcal{K}\}$$

and

<2> $$\mathcal{K} \subseteq \{h \in \mathcal{H} : \langle h - f_0, f_0 - f\rangle \ge 0\}$$

by the following steps.

(i) For all $a, b \in \mathcal{H}$ show that

$$\|a+b\|_2^2 + \|a-b\|_2^2 = 2\|a\|_2^2 + 2\|b\|_2^2 .$$

(ii) If $h_n \in \mathcal{K}$ are chosen to make $\delta_n := \|f - h_n\|_2 \to \delta$, show that $\{h_n : n \in \mathbb{N}\}$ is a Cauchy sequence in $\mathcal{H}$. Assuming completeness of $\mathcal{H}$, deduce that there exists some f_0 in $\mathcal{K}$ for which $\|h_n - f_0\|_2 \to 0$. Explain why f_0 satisfies <1>.

(iii) For all $h \in \mathcal{K}$ and all $t \in [0,1]$ explain why $\|f - (1-t)f_0 - th\|_2 \ge \delta$. Deduce <2>.