Statistics 330b/600b, Math 330b spring 2013
Homework # 9
Due: Thursday 11 April

Please attempt at least the starred problems.

*[1] (Neyman factorization theorem cf. UGMTP Example 5.31) Suppose $\mathbb{P}$ and $\mathbb{P}_\theta$, for $\theta \in \Theta$, are probability measures defined on a sigma-field $\mathcal{F}$, for some index set Θ. Suppose also that $\mathcal{G}$ is a sub-sigma-field of $\mathcal{F}$ and that there exist versions of densities

$$\frac{d\mathbb{P}_\theta}{d\mathbb{P}} = g_\theta(\omega)h(\omega) \qquad \text{with } g_\theta \in \mathcal{M}^+(\Omega, \mathcal{G}) \text{ for each } \theta$$

for a fixed $h \in \mathcal{M}^+(\Omega, \mathcal{F})$ that doesn't depend on θ. Define H to be a version of $\mathbb{P}_{\mathcal{G}} h$. [That is, choose one H from the $\mathbb{P}$-equivalence class of possibilities.]

(i) Show that $\mathbb{P}_\theta\{H \in B\} = \mathbb{P} g_\theta(\omega)\{H \in B\}H$ for each θ and each $B \in \mathcal{B}(\overline{\mathbb{R}})$.

(ii) Deduce that $\mathbb{P}_\theta\{H = 0\} = 0 = \mathbb{P}_\theta\{H = \infty\}$ for each θ.

(iii) For each X in $\mathcal{M}^+(\Omega, \mathcal{F})$ and some fixed choice of $\mathbb{P}_{\mathcal{G}}(Xh)$ define

$$Y(\omega) = \frac{\mathbb{P}_{\mathcal{G}}(Xh)}{H}\{0 < H < \infty\}.$$

Show that $\mathbb{P}_\theta(X \mid \mathcal{G}) = Y$ a.e. $[\mathbb{P}_\theta]$ for every θ.

[2] Suppose g_1 and g_2 are maps from $(\mathcal{X}, \mathcal{A})$ to $(\mathcal{Y}, \mathcal{B})$, with product-measurable graphs. Define $\psi_i(x) = (x, g_i(x))$, a measurable map from $\mathcal{X}$ into $\mathcal{X} \times \mathcal{Y}$. Let $P_i = \psi_i(\mu_i)$, for probability measures μ_i on $\mathcal{A}$. Let $P = \alpha_1 P_1 + \alpha_2 P_2$, for constants $\alpha_i > 0$ with $\alpha_1 + \alpha_2 = 1$. Let X denote the coordinate map from $\mathcal{X} \times \mathcal{Y}$ onto the $\mathcal{X}$ space. Show that the conditional probability distribution $P_x = P(\cdot \mid X = x)$ concentrates on the points $(x, g_1(x))$ and $(x, g_2(x))$. Find the conditional probabilities assigned to each point. Hint: Consider the density of μ_i with respect to $\alpha_1\mu_1 + \alpha_2\mu_2$.

[3] Suppose $X \in \mathcal{L}^1(\Omega, \mathcal{F}, \mathbb{P})$ and $\mathcal{G}$ is a sub-σ-field of $\mathcal{F}$ containing all $\mathbb{P}$-negligible sets. Show that X is $\mathcal{G}$-measurable if and only if $\mathbb{P}(XW) = 0$ for every bounded random variable W with $\mathbb{P}_{\mathcal{G}} W = 0$ almost surely by the following steps.

> **Remark.** Compare with the corresponding statement for random variables that are square integrable: $Z \in \mathcal{L}^2(\mathcal{G})$ if and only if it is orthogonal to every square integrable W that is orthogonal to $\mathcal{L}^2(\mathcal{G})$.

(i) Suppose t is a real number t for which $\mathbb{P}\{X = t\} = 0$. Define $Z_t = \mathbb{P}_{\mathcal{G}}\{X > t\}$ and $W_t = \{X > t\} - Z_t$. Show that $(X - t)W_t \geq 0$ almost surely.

(ii) Explain why $\mathbb{P}\left((X - t)W_t\right) = 0$. Deduce that $(X - t)W_t = 0$ almost surely.

(iii) Deduce that $\{X > t\} \in \mathcal{G}$ for every t with $\mathbb{P}\{X = t\} = 0$. Conclude that X is $\mathcal{G}$-measurable.