Appendix A

Measures and integrals

SECTION 1 introduces a method for constructing a measure by inner approximation, starting from a set function defined on a lattice of sets.

SECTION 2 defines a "tightness" property, which ensures that a set function has an extension to a finitely additive measure on a field determined by the class of approximating sets.

SECTION 3 defines a "sigma-smoothness" property, which ensures that a tight set function has an extension to a countably additive measure on a sigma-field.

SECTION 4 shows how to extend a tight, sigma-smooth set function from a lattice to its closure under countable intersections.

SECTION 5 constructs Lebesgue measure on Euclidean space.

SECTION 6 proves a general form of the Riesz representation theorem, which expresses linear functionals on cones of functions as integrals with respect to countably additive measures.

1. Measures and inner measure

Recall the definition of a ***countably additive measure*** on ***sigma-field***. A sigma-field $\mathcal{A}$ on a set $\mathcal{X}$ is a class of subsets of $\mathcal{X}$ with the following properties.

(SF_1) *The empty set* $\emptyset$ *and the whole space* $\mathcal{X}$ *both belong to* $\mathcal{A}$.

(SF_2) *If* A *belongs to* $\mathcal{A}$ *then so does its complement* A^c.

(SF_3) *For countable* $\{A_i : i \in \mathbb{N}\} \subseteq \mathcal{A}$, *both* $\cup_i A_i$ *and* $\cap_i A_i$ *are also in* $\mathcal{A}$.

A function μ defined on the sigma-field $\mathcal{A}$ is called a countably additive (nonnegative) measure if it has the following properties.

(M_1) $\mu\emptyset = 0 \le \mu A \le \infty$ *for each* A *in* $\mathcal{A}$.

(M_2) $\mu\left(\cup_i A_i\right) = \sum_i \mu A_i$ *for sequences* $\{A_i : i \in \mathbb{N}\}$ *of pairwise disjoint sets from* $\mathcal{A}$.

If property SF_3 is weakened to require stability only under finite unions and intersections, the class is called a ***field***. If property M_2 is weakened to hold only for disjoint unions of finitely many sets from $\mathcal{A}$, the set function is called a ***finitely additive measure***.

Where do measures come from? Typically one starts from a nonnegative real-valued set-function μ defined on a small class of sets $\mathcal{K}_0$, then extends to a sigma-field $\mathcal{A}$ containing $\mathcal{K}_0$. One must at least assume "measure-like" properties for μ on $\mathcal{K}_0$ if such an extension is to be possible. At a bare minimum,

(M_0) *μ is an increasing map from $\mathcal{K}_0$ into $\mathbb{R}^+$ for which $\mu\emptyset = 0$.*

Note that we need $\mathcal{K}_0$ to contain $\emptyset$ for M_0 to make sense. I will assume that M_0 holds thoughout this Appendix. As a convenient reminder, I will also reserve the name ***set function on*** $\mathcal{K}_0$ for those μ that satisfy M_0.

The extension can proceed by various approximation arguments. In the first three Sections of this Appendix, I will describe only the method based on approximation of sets from inside. Although not entirely traditional, the method has the advantage that it leads to measures with a useful approximation property called $\mathcal{K}_0$-***regularity***:

$$\mu A = \sup\{\mu K : A \supseteq K \in \mathcal{K}_0\} \qquad \text{for each } A \text{ in } \mathcal{A}.$$

REMARK. When $\mathcal{K}$ consists of compact sets, a measure with the inner regularity property is often called a ***Radon measure***.

The desired regularity property makes it clear how the extension of μ must be constructed, namely, by means of the ***inner measure*** μ_*, defined for *every* subset A of $\mathcal{X}$ by $\mu_* A := \sup\{\mu K : A \supseteq K \in \mathcal{K}_0\}$.

In the business of building measures it pays to start small, imposing as few conditions on the initial domain $\mathcal{K}_0$ as possible. The conditions are neatly expressed by means of some picturesque terminology. Think of $\mathcal{X}$ as a large expanse of muddy lawn, and think of subsets of $\mathcal{X}$ as paving stones to lay on the ground, with overlaps permitted. Then a collection of subsets of $\mathcal{X}$ would be a ***paving*** for $\mathcal{X}$. The analogy might seem far-fetched, but it gives a concise way to describe properties of various classes of subsets. For example, a field is nothing but a $(\emptyset, \cup f, \cap f, {}^c)$ paving, meaning that it contains the empty set and is stable under the formation of finite unions ($\cup f$), finite intersections ($\cap f$), and complements (^{c}). A $(\emptyset, \cup c, \cap c, {}^c)$ paving is just another name for a sigma-field—the $\cup c$ and $\cap c$ denote countable unions and intersections. With inner approximations the natural assumption is that $\mathcal{K}_0$ be at least a $(\emptyset, \cup f, \cap f)$ paving—a ***lattice*** of subsets.

REMARK. Note well. A lattice is not assumed to be stable under differences or the taking of complements. Keep in mind the prime example, where $\mathcal{K}_0$ denotes the class of compact subsets of a (Hausdorff) topological space, such as the real line. Inner approximation by compact sets has turned out to be a *good thing* for probability theory.

For a general lattice $\mathcal{K}_0$, the role of the closed sets (remember $\underline{\text{f}}$ for $\underline{\text{f}}$ermé) is played by the class $\mathcal{F}(\mathcal{K}_0)$ of all subsets F for which $FK \in \mathcal{K}_0$ for every K in $\mathcal{K}_0$. (Of course, $\mathcal{K}_0 \subseteq \mathcal{F}(\mathcal{K}_0)$. The inclusion is proper if $\mathcal{X} \notin \mathcal{K}_0$.) The sigma-field $\mathcal{B}(\mathcal{K}_0)$ generated by $\mathcal{F}(\mathcal{K}_0)$ will play the role of the Borel sigma-field.

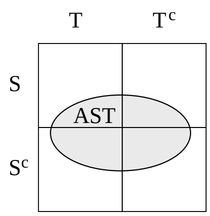
The first difficulty along the path leading to countably additive measures lies in the choice of the sigma-field $\mathcal{A}$, in order that the restriction of μ_* to $\mathcal{A}$ has the desired countable additivity properties. The Carathéodory splitting method identifies a suitable class of sets by means of an apparently weak substitute for the finite additivity property. Define $\mathcal{S}_0$ as the class of all subsets S of $\mathcal{X}$ for which

<1> $$\mu_* A = \mu_*\big(AS\big) + \mu_*\big(AS^c\big) \qquad \text{for all subsets } A \text{ of } \mathcal{X}.$$

If $A \in \mathcal{S}_0$ then μ_* adds the measures of the disjoint sets AS and AS^c correctly. As far as μ_* is concerned, S splits the set A "properly."

<2> **Lemma.** *The class $\mathcal{S}_0$ of all subsets S with the property <1> is a field. The restriction of μ_* to $\mathcal{S}_0$ is a finitely additive measure.*

Proof. Trivially $\mathcal{S}_0$ contains the empty set (because $\mu_*\emptyset = 0$) and it is stable under the formation of complements. To establish the field property it suffices to show that $\mathcal{S}_0$ is stable under finite intersections.



Suppose S and T belong to $\mathcal{S}_0$. Let A be an arbitrary subset of $\mathcal{X}$. Split A into two pieces using S, then split each of those two pieces using T. From the defining property of $\mathcal{S}_0$,

$$\begin{aligned}\mu_* A &= \mu_*(AS) + \mu_*\left(AS^c\right) \\ &= \mu_*(AST) + \mu_*\left(AST^c\right) + \mu_*\left(AS^cT\right) + \mu_*\left(AS^cT^c\right).\end{aligned}$$

Decompose $A(ST)^c$ similarly to see that the last three terms sum to $\mu_* A(ST)^c$. The intersection ST splits A correctly; the class $\mathcal{S}_0$ contains ST; the class is a field. If $ST = 0$, choose $A := S \cup T$ to show that the restriction of μ_* to $\mathcal{S}_0$ is finitely additive. □

At the moment there is no guarantee that $\mathcal{S}_0$ includes all the members of $\mathcal{K}_0$, let alone all the members of $\mathcal{B}(\mathcal{K}_0)$. In fact, the Lemma has nothing to do with the choice of μ and $\mathcal{K}_0$ beyond the fact that $\mu_*(\emptyset) = 0$. To ensure that $\mathcal{S}_0 \supseteq \mathcal{K}_0$ we must assume that μ has a property called $\mathcal{K}_0$-***tightness***, an analog of finite additivity that compensates for the fact that the difference of two $\mathcal{K}_0$ sets need not belong to $\mathcal{K}_0$. Section 2 explains $\mathcal{K}_0$-tightness. Section 3 adds the assumptions needed to make the restriction of μ_* to $\mathcal{S}_0$ a countable additivity measure.

2. Tightness

If $\mathcal{S}_0$ is to contain every member of $\mathcal{K}_0$, every set $K \in \mathcal{K}_0$ must split every set $K_1 \in \mathcal{K}_0$ properly, in the sense of Definition <1>,

$$\mu_*(K_1) = \mu_*(K_1 K) + \mu_*(K_1 \backslash K).$$

Writing K_0 for $K_1 K$, we then have the following property as a necessary condition for $\mathcal{K}_0 \subseteq \mathcal{S}_0$. It will turn out that the property is also sufficient.

<3> **Definition.** *Say that a set function μ on $\mathcal{K}_0$ is $\mathcal{K}_0$-tight if $\mu K_1 = \mu K_0 + \mu_*(K_1 \backslash K_0)$ for all pairs of sets in $\mathcal{K}_0$ with $K_1 \supseteq K_0$.*

The intuition is that there exists a set $K \in \mathcal{K}_0$ that almost fills out $K_1 \backslash K_0$, in the sense that $\mu K \approx \mu K_1 - \mu K_0$. More formally, for each $\epsilon > 0$ there exists a $K_\epsilon \in \mathcal{K}_0$ with $K_\epsilon \subseteq K_1 \backslash K_0$ and $\mu K_\epsilon > \mu K_1 - \mu K_0 - \epsilon$. As a convenient abbreviation, I will say that such a K_ϵ fills out the difference $K_1 \backslash K_0$ within an ϵ.

Tightness is as close as we come to having $\mathcal{K}_0$ stable under proper differences. It implies a weak additivity property: if K and H are disjoint members of $\mathcal{K}_0$ then $\mu(H \cup K) = \mu H + \mu K$, because the supremum in the definition of $\mu_*\big((H \cup K) \backslash K\big)$ is achieved by H. Additivity for disjoint $\mathcal{K}_0$-sets implies superadditivity for the inner measure,

<4> $$\mu_*(A \cup B) \geq \mu_* A + \mu_* B \qquad \text{for all disjoint } A \text{ and } B,$$

because the union of each inner approximating H for A and each inner approximating K for B is an inner approximating set for $A \cup B$. Tightness also gives us a way to relate $\mathcal{S}_0$ to $\mathcal{K}_0$.

<5> **Lemma.** *Let $\mathcal{K}_0$ be a $(\emptyset, \cup f, \cap f)$ paving, and μ be $\mathcal{K}_0$-tight set function. Then*

(i) $S \in \mathcal{S}_0$ if and only if $\mu K \leq \mu_* (KS) + \mu_* (K \backslash S)$ for all K in $\mathcal{K}_0$;

(ii) the field $\mathcal{S}_0$ contains the field generated by $\mathcal{F}(\mathcal{K}_0)$.

Proof. Take a supremum in (i) over all $K \subseteq A$ to get $\mu_* A \leq \mu_* (AS) + \mu_* (A \backslash S)$. The superadditivity property <4> gives the reverse inequality.

If $S \in \mathcal{F}(\mathcal{K}_0)$ and $K \in \mathcal{K}_0$, the pair $K_1 := K$ and $K_0 := KS$ are candidates for the tightness equality, $\mu K = \mu (KS) + \mu_* (K \backslash S)$, implying the inequality in (i). □

3. Countable additivity

Countable additivity ensures that measures are well behaved under countable limit operations. To fit with the lattice properties of $\mathcal{K}_0$, it is most convenient to insert countable additivity into the construction of measures via a limit requirement that has been called ***σ-smoothness*** in the literature. I will stick with that term, rather than invent a more descriptive term (such as σ-continuity from above), even though I feel that it conveys not quite the right image for a set function.

<6> **Definition.** *Say that μ is σ-smooth (along $\mathcal{K}_0$) at a set K in $\mathcal{K}_0$ if $\mu K_n \downarrow \mu K$ for every decreasing sequence of sets $\{K_n\}$ in $\mathcal{K}_0$ with intersection K.*

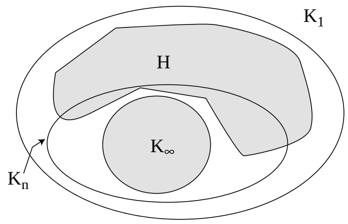
> REMARK. It is important that μ takes only (finite) real values for sets in $\mathcal{K}_0$. If λ is a countably additive measure on a sigma-field $\mathcal{A}$, and $A_n \downarrow A_\infty$ with all A_i in $\mathcal{A}$, then we need not have $\lambda A_n \downarrow \lambda A_\infty$ unless $\lambda A_n < \infty$ for some n, as shown by the example of Lebesgue measure with $A_n = [n, \infty)$ and $A_\infty = \emptyset$.

Notice that the definition concerns only those decreasing sequences in $\mathcal{K}_0$ for which $\bigcap_{n \in \mathbb{N}} K_n \in \mathcal{K}_0$. At the moment, there is no presumption that $\mathcal{K}_0$ be stable under countable intersections. As usual, the σ is to remind us of the restriction to *countable* families. There is a related property called τ-smoothness, which relaxes the assumption that there are only countably many K_n sets—see Problem [1].

Tightness simplifies the task of checking for σ-smoothness. The next proof is a good illustration of how one makes use of $\mathcal{K}_0$-tightness and the fact that μ_* has already been proven finitely additive on the field $\mathcal{S}_0$.

<7> **Lemma.** *If a $\mathcal{K}_0$-tight set function on a $(\emptyset, \cup f, \cap f)$ paving $\mathcal{K}_0$ is σ-smooth at $\emptyset$ then it is σ-smooth at every set in $\mathcal{K}_0$.*

Proof. Suppose $K_n \downarrow K_\infty$, with all K_i in $\mathcal{K}_0$. Find an $H \in \mathcal{K}_0$ that fills out the difference $K_1\backslash K_\infty$ within ϵ. Write L for $H \cup K_\infty$. Finite additivity of μ_* on $\mathcal{S}_0$ lets us break μK_n into the sum



$$\mu K_\infty + \mu\,(K_n H) + \mu_*\,(K_n\backslash L)\,.$$

The middle term decreases to zero as $n \to \infty$ because $K_n H \downarrow K_\infty H = \emptyset$. The last term is less than

$$\mu_*\,(K_1\backslash L) = \mu K_1 - \mu K_\infty - \mu H = \mu_*\,(K_1\backslash K_\infty) - \mu H,$$

which is less than ϵ, by construction. □

If $\mathcal{K}_0$ is a stable under countable intersections, the σ-smoothness property translates easily into countable additivity for μ_* as a set function on $\mathcal{S}_0$.

<8> **Theorem.** *Let $\mathcal{K}_0$ be a lattice of subsets of $\mathcal{X}$ that is stable under countable intersections, that is, a $(\emptyset, \cup f, \cap c)$ paving. Let μ be a $\mathcal{K}_0$-tight set function on a $\mathcal{K}_0$, with associated inner measure $\mu_* A := \sup\{\mu K : A \supseteq K \in \mathcal{K}_0\}$. Suppose μ is σ-smooth at $\emptyset$ (along $\mathcal{K}_0$). Then*

(i) the class

$$\mathcal{S}_0 := \{S \subseteq \mathcal{X} : \mu K \le \mu_*(KS) + \mu_*(K\backslash S) \text{ for all } K \text{ in } \mathcal{K}_0\}$$

is a sigma-field on $\mathcal{X}$;

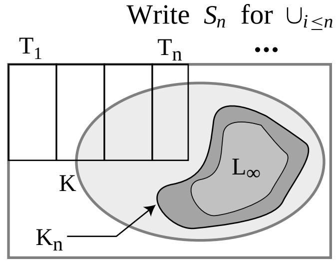
(ii) $\mathcal{S}_0 \supseteq \mathcal{B}(\mathcal{K}_0)$, the sigma-field generated by $\mathcal{F}(\mathcal{K}_0)$;

(iii) the restriction of μ_* to $\mathcal{S}_0$ is a $\mathcal{K}_0$-regular, countably additive measure;

(iv) $\mathcal{S}_0$ is complete: if $S_1 \supseteq B \supseteq S_0$ with $S_i \in \mathcal{S}_0$ and $\mu_*(S_1\backslash S_0) = 0$ then $B \in \mathcal{S}_0$.

Proof. From Lemma <5>, we know that $\mathcal{S}_0$ is a field that contains $\mathcal{F}(\mathcal{K}_0)$. To prove (i) and (ii), it suffices to show that the union $S := \cup_{i\in\mathbb{N}} T_i$ of a sequence of sets in $\mathcal{S}_0$ also belongs to $\mathcal{S}_0$, by establishing the inequality $\mu K \le \mu_*\,(KS) + \mu_*\,(K\backslash S)$, for each choice of K in $\mathcal{K}_0$.

Write S_n for $\cup_{i\le n} T_i$. For a fixed $\epsilon > 0$ and each i, choose a $\mathcal{K}_0$-subset K_i of $K\backslash S_i$ for which $\mu K_i > \mu_*\,(K\backslash S_i) - \epsilon/2^i$. Define $L_n := \cap_{i\le n} K_i$. Then, by the finite additivity of μ_* on $\mathcal{S}_0$,



$$\mu_*(K\backslash S_n) - \mu L_n \le \textstyle\sum_{i\le n}\big(\mu_*(K\backslash S_i) - \mu K_i\big) < \epsilon.$$

The sequence of sets $\{L_n\}$ decreases to a $\mathcal{K}_0$-subset L_∞ of $K\backslash S$. By the σ-smoothness at L_∞ we have $\mu L_n \le \mu L_\infty + \epsilon \le \mu_*\,(K\backslash S) + \epsilon$, for n large enough, which gives $\mu_*\,(K\backslash S_n) \le \mu L_n + \epsilon \le \mu_*\,(K\backslash S) + 2\epsilon$, whence

$$\begin{aligned}\mu K &= \mu_*\,(KS_n) + \mu_*\,(K\backslash S_n) \qquad \text{because } S_n \in \mathcal{S}_0\\ &\le \mu_*\,(KS) + \mu_*\,(K\backslash S) + 2\epsilon.\end{aligned}$$

It follows that $S \in \mathcal{S}_0$.

When $K \subseteq S$, the inequality $\mu K \le \mu_*\,(KS_n) + \mu_*\,(K\backslash S) + 2\epsilon$ and the finite additivity of μ_* on $\mathcal{S}_0$ imply $\mu K \le \sum_{i\le n} \mu_*\,(KT_i) + 2\epsilon$. Take the supremum over all $\mathcal{K}_0$-subsets of S, let n tend to infinity, then ϵ tend to zero, to deduce that $\mu_* S \le \sum_{i\in\mathbb{N}} \mu_* T_i$. The reverse inequality follows from the superadditivity property <4>. The set function μ_* is countably additive on the the sigma-field $\mathcal{S}_0$.

For (iv), note that $\mu K = \mu_*(KS_0) + \mu_*(K\backslash S_0)$, which is smaller than

$$\mu_*(KB) + \mu_*(K\backslash S_1) + \mu_*(KS_1S_0^c) \le \mu_*(KB) + \mu_*(K\backslash B) + 0,$$

□ for every K in $\mathcal{K}_0$.

In one particularly important case we get σ-smoothness for free, without any extra assumptions on the set function μ. A paving $\mathcal{K}_0$ is said to be ***compact*** (in the sense of Marczewski 1953) if: to each countable collection $\{K_i : i \in \mathbb{N}\}$ of sets from $\mathcal{K}_0$ with $\cap_{i\in\mathbb{N}} K_i = \emptyset$ there is some finite n for which $\cap_{i\le n} K_i = \emptyset$. In particular, if $K_i \downarrow \emptyset$ then $K_n = \emptyset$ for some n. For such a paving, the σ-smoothness property places no constraint on μ beyond the standard assumption that $\mu\emptyset = 0$.

<9> **Example.** Let $\mathcal{K}_0$ be a collection of closed, compact subsets of a topological space $\mathcal{X}$. Suppose $\{K_\alpha : \alpha \in A\}$ is a subcollection of $\mathcal{K}_0$ for which $\cap_{\alpha\in A} K_\alpha = \emptyset$. Arbitrarily choose an α_0 from A. The collection of open sets $G_\alpha := K_\alpha^c$ for $\alpha \in A$ covers the compact set K_{α_0}. By the definition of compactness, there exists a finite subcover. That is, for some $\alpha_1, \dots, \alpha_m$ we have $K_{\alpha_0} \subseteq \cup_{i=1}^m G_{\alpha_i} = \left(\cap_{i=1}^m K_{\alpha_i}\right)^c$. Thus □ $\cap_{i=0}^m K_{\alpha_i} = \emptyset$. In particular, $\mathcal{K}_0$ is also compact in the Marczewski sense.

REMARK. Notice that the Marczewski concept involves only countable subcollections of $\mathcal{K}_0$, whereas the topological analog from Example <9> applies to arbitrary subcollections. The stronger property turns out to be useful for proving τ-smoothness, a property stronger than σ-smoothness. See Problem [1] for the definition of τ-smoothness.

4. Extension to the $\cap c$-closure

If $\mathcal{K}_0$ is not stable under countable intersections, σ-smoothness is not quite enough to make μ_* countably additive on $\mathcal{S}_0$. We must instead work with a slightly richer approximating class, derived from $\mathcal{K}_0$ by taking its ***$\cap c$-closure***: the class $\mathcal{K}$ of all intersections of countable subcollections from $\mathcal{K}_0$. Clearly $\mathcal{K}$ is stable under countable intersections. Also stability under finite unions is preserved, because

$$(\cap_{i\in\mathbb{N}} H_i) \cup \left(\cap_{j\in\mathbb{N}} K_j\right) = \cap_{i,j\in\mathbb{N}\times\mathbb{N}} \left(H_i \cup K_j\right),$$

a countable intersection of sets from $\mathcal{K}_0$. Note also that if $\mathcal{K}_0$ is a compact paving then so is $\mathcal{K}$.

The next Lemma shows that the natural extension of μ to a set function on $\mathcal{K}$ inherits the desirable σ-smoothness and tightness properties.

<10> **Lemma.** *Let μ be a $\mathcal{K}_0$-tight set function on a $(\emptyset, \cup f, \cap f)$ paving $\mathcal{K}_0$, which is σ-smooth along $\mathcal{K}_0$ at $\emptyset$. Then the extension $\tilde{\mu}$ of μ to the $\cap c$-closure $\mathcal{K}$, defined by*

<11> $$\tilde{\mu} H := \inf\{\mu K : H \subseteq K \in \mathcal{K}_0\} \qquad \text{for } H \in \mathcal{K},$$

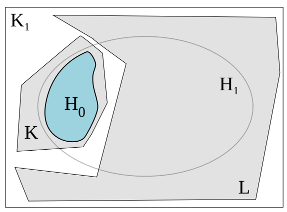
is $\mathcal{K}$-tight and σ-smooth (along $\mathcal{K}$) at $\emptyset$.

Proof. Lemma <7> gives us a simpler expression for $\tilde{\mu}$. If $\{K_n : n \in \mathbb{N}\} \subseteq \mathcal{K}_0$ and $K_n \downarrow L \in \mathcal{K}$, then $\tilde{\mu}L = \inf_n \mu K_n$ because, for each $\mathcal{K}_0$-subset K with $K \supseteq L$,

$$\tilde{\mu}K_n \le \mu(K_n \cup K) \downarrow \mu K \qquad \text{by } \sigma\text{-smoothness of } \mu \text{ at } K.$$

The σ-smoothness at $\emptyset$ is easy. If $H_n := \bigcap_{j\in\mathbb{N}} K_{nj} \in \mathcal{K}$ and $H_n \downarrow \emptyset$ then the sets $K_n := \bigcap_{i\le n, j\le n} K_{ij}$ belong to $\mathcal{K}_0$, and $H_n = H_1 H_2 \dots H_n \subseteq K_n \downarrow \emptyset$. It follows that $\tilde{\mu}H_n \le \mu K_n \downarrow 0$.

The $\mathcal{K}$-tightness is slightly trickier. Suppose $H_1 \supseteq H_0$, with both sets in $\mathcal{K}$. Let $\{K_n\}$ be a decreasing sequence of sets in $\mathcal{K}_0$ with intersection H_1. For a fixed $\epsilon > 0$, choose a K in $\mathcal{K}_0$ with $K \supseteq H_0$ and $\mu K < \tilde{\mu}H_0 + \epsilon$. With no loss of generality we may assume that $K \subseteq K_1$. Invoke $\mathcal{K}_0$-tightness to find a $\mathcal{K}_0$-subset L of $K_1 \backslash K$ for which $\mu L > \mu K_1 - \mu K - \epsilon > \mu K_1 - \tilde{\mu}H_0 - 2\epsilon$. Notice that the sequence $\{LK_n\}$



decreases to LH_1, a $\mathcal{K}$-subset of $H_1\backslash K \subseteq H_1\backslash H_0$. The finite additivity of μ_*, when restricted to $\mathcal{S}_0$, gives

$$\begin{aligned}\mu(LK_n) &= \mu L + \mu K_n - \mu(L \cup K_n)\\ &> (\mu K_1 - \tilde{\mu}H_0 - 2\epsilon) + \mu K_n - \mu K_1\\ &\to -\tilde{\mu}H_0 - 2\epsilon + \tilde{\mu}H_1 \qquad \text{as } n \to \infty.\end{aligned}$$

□ Thus $\tilde{\mu}(LH_1) \ge \tilde{\mu}H_1 - \tilde{\mu}H_0 - 2\epsilon$, as required for $\mathcal{K}$-tightness.

REMARK. It is helpful, but not essential, to have a different symbol for the extension of μ to a larger domain while we are establishing properties for that extension. For example, it reminds us not to assume that $\tilde{\mu}$ has the same properties as μ before we have proved as much. Once the result is proven, the $\tilde{\mu}$ has served its purpose, and it can then safely be replaced by μ.

A similar argument might be made about the distinction between $\mathcal{K}_0$ and $\mathcal{K}$, but there is some virtue in retaining the subscript as a reminder than $\mathcal{K}_0$ is assumed stable only under finite intersections.

Together, Theorem <8> and Lemma <10> give a highly useful extension theorem for set functions defined initially on a lattice of subsets.

<12> **Theorem.** *Let $\mathcal{K}_0$ be a $(\emptyset, \cup f, \cap f)$ paving of subsets of $\mathcal{X}$, and let $\mathcal{K}$ denote its ∩c-closure. Let $\mu : \mathcal{K}_0 \to \mathbb{R}^+$ be a $\mathcal{K}_0$-tight set function that is sigma-smooth along $\mathcal{K}_0$ at $\emptyset$. Then μ has a unique extension to a complete, $\mathcal{K}$-regular, countably additive measure on a sigma-field $\mathcal{S}$, defined by*

$$\mu K := \inf\{\mu K_0 : K \subseteq K_0 \in \mathcal{K}_0\} \qquad \text{for } K \in \mathcal{K},$$
$$\mu S := \sup\{\mu K : S \supseteq K \in \mathcal{K}\} \qquad \text{for } S \in \mathcal{S}.$$

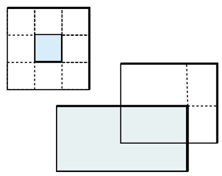
The sigma-field $\mathcal{S}$ contains all sets F for which $FK \in \mathcal{K}$ for all K in $\mathcal{K}$. In particular, $\mathcal{S} \supseteq \mathcal{K} \supseteq \mathcal{K}_0$.

REMARK. Remember: σ-smoothness is automatic if $\mathcal{K}_0$ is a compact paving.

5. Lebesgue measure

There are several ways in which to construct Lebesgue measure on $\mathbb{R}^k$. The following method for $\mathbb{R}^2$ is easily extended to other dimensions.

Take $\mathcal{K}_0$ to consist of all finite unions of semi-open rectangles $(\alpha_1, \beta_1] \otimes (\alpha_2, \beta_2]$. Each difference of two semi-open rectangles can be written as a disjoint union of at most eight similar rectangles. As a consequence, every member of $\mathcal{K}_0$ has a representation as a finite union of *disjoint* semi-open rectangles, and $\mathcal{K}_0$ is stable under the formation of differences. The initial definition of Lebesgue measure $\mathfrak{m}$, as a set function on $\mathcal{K}_0$, might seem obvious—add up the areas of the disjoint rectangles. It is a surprisingly tricky exercise to prove rigorously that $\mathfrak{m}$ is well defined and finitely additive on $\mathcal{K}_0$.



REMARK. The corresponding argument is much easier in one dimension. It is, perhaps, simpler to consider only that case, then obtain Lebesgue measure in higher dimensions as a completion of products of one-dimensional Lebesgue measures.

The $\mathcal{K}_0$-tightness of $\mathfrak{m}$ is trivial, because $\mathcal{K}_0$ is stable under differences: if $K_1 \supseteq K_0$, with both sets in $\mathcal{K}_0$, then $K_1 \backslash K_0 \in \mathcal{K}_0$ and $\mathfrak{m}K_1 - \mathfrak{m}K_0 = \mathfrak{m}(K_1 \backslash K_0)$.

To establish σ-smoothness, consider a decreasing sequence $\{K_n\}$ with empty intersection. Fix $\epsilon > 0$. If we shrink each component rectangle of K_n by a small enough amount we obtain a set L_n in $\mathcal{K}_0$ whose closure $\bar{L}_n$ is a compact subset of K_n and for which $\mathfrak{m}(K_n \backslash L_n) < \epsilon/2^n$. The family of compact sets $\{\bar{L}_n : n = 1, 2, \ldots\}$ has empty intersection. For some finite N we must have $\cap_{i \le N} \bar{L}_i = \emptyset$, so that

$$\mathfrak{m}K_N \le \mathfrak{m}\left(\cap_{i \le N} L_i\right) + \sum_{i \le N} \mathfrak{m}(K_i \backslash L_i) \le 0 + \sum_{i \le N} \epsilon/2^i.$$

It follows that $\mathfrak{m}K_n$ tends to zero as n tends to infinity. The finitely additive measure $\mathfrak{m}$ is $\mathcal{K}_0$-smooth at $\emptyset$. By Theorem <12>, it extends to a $\mathcal{K}$-regular, countably additive measure on $\mathcal{S}$, a sigma-field that contains all the sets in $\mathcal{F}(\mathcal{K})$.

You should convince yourself that $\mathcal{K}$, the $\cap c$-closure of $\mathcal{K}_0$, contains all compact subsets of $\mathbb{R}^2$, and $\mathcal{F}(\mathcal{K})$ contains all closed subsets. The sigma-field $\mathcal{S}$ is complete and contains the Borel sigma-field $\mathcal{B}(\mathbb{R}^2)$. In fact $\mathcal{S}$ is the Lebesgue sigma-field, the closure of the Borel sigma-field.

6. Integral representations

Throughout the book I have made heavy use of the fact that there is a one-to-one correspondence (via integrals) between measures and increasing linear functionals on $\mathcal{M}^+$ with the Monotone Convergence property. Occasionally (as in Sections 4.8 and 7.5), I needed an analogous correspondence for functionals on a subcone of $\mathcal{M}^+$. The methods from Sections 1, 2, and 3 can be used to construct measures representing such functionals if the subcone is stable under lattice-like operations.

<13> **Definition.** *Call a collection $\mathcal{H}^+$ of nonnegative real functions on a set $\mathcal{X}$ a* ***lattice cone*** *if it has the following properties. For h, h_1 and h_2 in $\mathcal{H}^+$, and α_1 and α_2 in $\mathbb{R}^+$:*

(H_1) $\alpha_1 h_1 + \alpha_2 h_2$ *belongs to* $\mathcal{H}^+$*;*

(H_2) $h_1 \backslash h_2 := (h_1 - h_2)^+$ *belongs to* $\mathcal{H}^+$*;*

(H_3) the pointwise minimum $h_1 \wedge h_2$ *and maximum* $h_1 \vee h_2$ *belong to* $\mathcal{H}^+$*;*

(H_4) $h \wedge 1$ *belongs to* $\mathcal{H}^+$*.*

The best example of a lattice cone to keep in mind is the class $\mathcal{C}_0^+(\mathbb{R}^k)$ of all nonnegative, continuous functions with compact support on some Euclidean space $\mathbb{R}^k$.

REMARK. By taking the positive part of the difference in H_2, we keep the function nonnegative. Properties H_1 and H_2 are what one would get by taking the collection of all positive parts of members of a vector space of functions. Property H_4 is sometimes called ***Stone's condition***. It is slightly weaker than an assumption that the constant function 1 should belong to $\mathcal{H}^+$. Notice that the cone $\mathcal{C}_0^+(\mathbb{R}^k)$ satisfies H_4, but it does not contain nonzero constants. Nevertheless, if $h \in \mathcal{H}^+$ and α is a positive constant then the function $(h-\alpha)^+ = (h - \alpha(1 \wedge h/\alpha))^+$ belongs to $\mathcal{H}^+$.

<14> **Definition.** *Say that a map $T : \mathcal{H}^+ \mapsto \mathbb{R}^+$ is an **increasing linear functional** if, for h_1, h_2 in $\mathcal{H}^+$, and α_1, α_2 in $\mathbb{R}^+$:*

(T_1) $T(\alpha_1 h_1 + \alpha_2 h_2) = \alpha_1 T h_1 + \alpha_2 T h_2$;

(T_2) $T h_1 \le T h_2$ *if* $h_1 \le h_2$ *pointwise.*

*Call the functional σ-**smooth at** 0 if*

(T_3) $T h_n \downarrow 0$ *whenever the sequence $\{h_n\}$ in $\mathcal{H}^+$ decreases pointwise to zero.*

*Say that T has the **truncation property** if*

(T_4) $T(h \wedge n) \to Th$ *as $n \to \infty$, for each h in $\mathcal{H}^+$.*

REMARK. For an increasing linear functional, T_3 is equivalent to an apparently stronger property,

(T_3') if $h_n \downarrow h_\infty$ with all h_i in $\mathcal{H}^+$ then $T h_n \downarrow T h_\infty$,

because $T h_n \le T h_\infty + T(h_n \backslash h_\infty) \downarrow T h_\infty + 0$. Property T_4 will allow us to reduce the representation of arbitrary members of $\mathcal{H}^+$ as integrals to the representation for bounded functions in $\mathcal{H}^+$.

If μ is a countably additive measure on a sigma-field $\mathcal{A}$, and all the functions in $\mathcal{H}^+$ are μ-integrable, then the $Th := \mu h$ defines a functional on $\mathcal{H}^+$ satisfying T_1 through T_4. The converse problem—find a μ to represent a given functional T—is called the ***integral representation problem***. Theorem <8> will provide a solution to the problem in some generality.

Let $\mathcal{K}_0$ denote the class of all sets K for which there exists a countable subfamily of $\mathcal{H}^+$ with pointwise infimum equal to (the indicator function of) K. Equivalently, by virtue of H_3, there is a decreasing sequence in $\mathcal{H}^+$ converging pointwise to K. It is easy to show that $\mathcal{K}_0$ is a $(\emptyset, \cup f, \cap c)$-paving of subsets for $\mathcal{X}$. Moreover, as the next Lemma shows, the functions in $\mathcal{H}^+$ are related to $\mathcal{K}_0$ and $\mathcal{F}(\mathcal{K}_0)$ in much the same way that nonnegative, continuous functions with compact support in $\mathbb{R}^k$ are related to compact and closed sets.

<15> **Lemma.** *For each h in $\mathcal{H}^+$ and each nonnegative constant α,*

(i) $\{h \ge \alpha\} \in \mathcal{K}_0$ *if* $\alpha > 0$, and *(ii)* $\{h \le \alpha\} \in \mathcal{F}(\mathcal{K}_0)$.

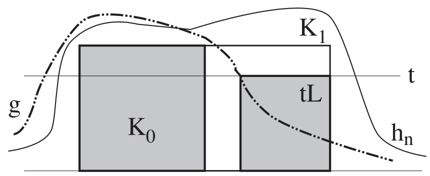
Proof. For (i), note that $\{h \ge \alpha\} = \inf_{n\in\mathbb{N}} \left(1 \wedge n \left(h - \alpha + n^{-1}\right)^+\right)$, a pointwise infimum of a sequence of functions in $\mathcal{H}^+$. For (ii), for a given K in $\mathcal{K}_0$, find a sequence $\{h_n : n \in \mathbb{N}\} \subseteq \mathcal{H}^+$ that decreases to K. Then note that $K\{h \le \alpha\} = \inf_n h_n \backslash (nh - n\alpha)^+$, a set that must therefore belong to $\mathcal{K}_0$. □

<16> **Theorem.** *Let $\mathcal{H}^+$ be a lattice cone of functions, satisfying requirements H_1 through H_4, and T be an increasing, linear functional on $\mathcal{H}^+$ satisfying conditions T_1 through T_4. Then the set function defined on $\mathcal{K}_0$ by $\mu K := \inf\{Th : K \le h \in \mathcal{H}^+\}$ is $\mathcal{K}_0$-tight and σ-smooth along $\mathcal{K}_0$ at $\emptyset$. Its extension to a $\mathcal{K}_0$-regular measure on $\mathcal{B}(\mathcal{K}_0)$ represents the functional, that is, $Th = \mu h$ for all h in $\mathcal{H}^+$. There is only one $\mathcal{K}_0$-regular measure on $\mathcal{B}(\mathcal{K}_0)$ whose integral represents T.*

> REMARK. Notice that we can replace the infimum in the definition of μ by an infimum along any decreasing sequence $\{h_n\}$ in $\mathcal{H}^+$ with pointwise limit K. For if $K \le h \in \mathcal{H}^+$, then $\inf_n Th_n \le \inf_n T(h_n \vee h) = Th$, by T_2 and T_3'.

Proof. We must prove that μ is σ-smooth along $\mathcal{K}_0$ at $\emptyset$ and $\mathcal{K}_0$-tight; and then prove that $Th \ge \mu h$ and $Th \le \mu h$ for every h in $\mathcal{H}^+$.

σ-smoothness: Suppose $K_n \in \mathcal{K}_0$ and $K_n \downarrow \emptyset$. Express K_n as a pointwise infimum of functions $\{h_{n,i}\}$ in $\mathcal{H}^+$. Write h_n for $\inf_{m\le n,\, i\le n} h_{m,i}$. Then $K_n \le h_n \downarrow 0$, and hence $\mu K_n \le Th_n \downarrow 0$ by the σ-smoothness for T and the definition of μ.



$\mathcal{K}_0$-tightness: Consider sets $K_1 \supseteq K_0$ in $\mathcal{K}_0$. Choose $\mathcal{H}^+$ functions $g \ge K_0$ and $h_n \downarrow K_1$ and fix a positive constant $t < 1$. The $\mathcal{H}^+$-function $g_n := (h_n - n(g\backslash t))^+$ decreases pointwise to the set $L := K_1\{g \le t\} \subseteq K_1\backslash K_0$. Also, it is trivially true that $g \ge tK_1\{g > t\}$. From the inequality $g_n + g \ge tK_1$ we get $\mu K_1 \le T(g_n + g)/t$, because $(g_n + g)/t$ is one of the $\mathcal{H}^+$-functions that enters into the definition of μK_1. Let n tend to infinity, take an infimum over all $g \ge K_0$, then let t increase to 1, to deduce that $\mu K_1 \le \mu L + \mu K_0$, as required for $\mathcal{K}_0$-tightness.

By Theorem <8>, the set function μ extends to a $\mathcal{K}_0$-regular measure on $\mathcal{B}(\mathcal{K}_0)$

Inequality $Th \ge \mu h$: Suppose $h \ge u := \sum_{j=1}^k \alpha_j A_j \in \mathcal{M}^+_{\text{simple}}$. We need to show that $Th \ge \mu u := \sum_j \alpha_j \mu A_j$. We may assume that the $\mathcal{A}$-measurable sets A_j are disjoint. Choose $\mathcal{K}_0$ sets $K_j \subseteq A_j$, thereby defining another simple function $v := \sum_{j=1}^k \alpha_j K_j \le u$. Find sequences h_{nj} from $\mathcal{H}^+$ with $h_{nj} \downarrow \alpha_j K_j$, so that $\sum_j Th_{nj} \downarrow \sum_j \alpha_j \mu K_j = \mu v$. With no loss of generality, assume $h \ge h_{nj}$ for all n and j. Then we have a pointwise bound, $\sum_j h_{nj} \le h + \sum_{i<j} h_{ni} \wedge h_{nj}$, because $\max_j h_{nj} \le h$ and each of the smaller h_{nj} summands must appear in the last sum. Thus

$$\mu v := \sum_j \alpha_j \mu K_j \le \sum_j Th_{nj} \le Th + \sum_{i<j} T\left(h_{ni} \wedge h_{nj}\right).$$

As n tends to infinity, $h_{ni} \wedge h_{nj} \downarrow K_i K_j = \emptyset$. By σ-smoothness of T, the right-hand side decreases to Th, leaving $\mu v \le Th$. Take the supremum over all $K_j \subseteq A_j$, then take the supremum over all $u \le h$, to deduce that $\mu h \le Th$.

Inequality $Th \le \mu h$: Invoke property T_4 to reduce to the case of a bounded h. For a fixed $\epsilon > 0$, approximate h by a simple function $s_\epsilon := \epsilon \sum_{i=1}^N \{h \ge i\epsilon\}$, with steps of size ϵ. Here N is a fixed value large enough to make $N\epsilon$ an upper bound for h. Notice that $\{h \ge i\epsilon\} \in \mathcal{K}_0$, by Lemma <15>. Find sequences h_{ni} from $\mathcal{H}^+$ with $h_{ni} \downarrow \{h \ge i\epsilon\}$. Then we have

$$s_\epsilon \le h \le (h \wedge \epsilon) + s_\epsilon \le (h \wedge \epsilon) + \epsilon \sum_{i=1}^N h_{ni},$$

from which it follows that

$$\begin{aligned} Th &\le T(h \wedge \epsilon) + \epsilon \sum_{i=1}^N Th_{ni} \\ &\to T(h \wedge \epsilon) + \epsilon \sum_{i=1}^N \mu\{h \ge i\epsilon\} \qquad \text{as } n \to \infty \\ &\le T(h \wedge \epsilon) + \mu h \qquad \text{because } \epsilon \sum_{i=1}^N \{h \ge i\epsilon\} = s_\epsilon \le h \\ &\to \mu h \qquad \text{as } \epsilon \to 0, \text{ by } \sigma\text{-smoothness of } T. \end{aligned}$$

Uniqueness: Let ν be another $\mathcal{K}_0$-regular representing measure. If $h_n \downarrow K \in \mathcal{K}_0$, and $h_n \in \mathcal{H}^+$, then $\mu K = \lim_n \mu h_n = \lim_n Th_n = \lim_n \nu h_n = \nu K$. Regularity extends the equality to all sets in $\mathcal{B}(\mathcal{K}_0)$. □

<17> **Example.** Let $\mathcal{H}^+$ equal $\mathcal{C}_0^+(\mathcal{X})$, the cone of all nonnegative, continuous functions with compact support on a locally compact, Hausdorff space $\mathcal{X}$. For example, $\mathcal{X}$ might be $\mathbb{R}^k$. Let T be an increasing linear functional on $\mathcal{C}_0^+(\mathcal{X})$.

Property T_4 holds for the trivial reason that each member of $\mathcal{C}_0^+(\mathcal{X})$ is bounded. Property T_3 is automatic, for a less trivial reason. Suppose $h_n \downarrow 0$. Without loss of generality, $K \ge h_1$ for some compact K. Choose h in $\mathcal{C}_0^+(\mathcal{X})$ with $h \ge K$. For fixed $\epsilon > 0$, the union of the open sets $\{h_n < \epsilon\}$ covers K. For some finite N, the set $\{h_N < \epsilon\}$ contains K, in which case $h_N \le \epsilon K \le \epsilon h$, and $Th_N \le \epsilon Th$. The σ-smoothness follows.

The functional T has a representation $Th = \mu h$ on $\mathcal{C}_0^+(\mathcal{X})$, for a $\mathcal{K}_0$-regular measure μ. The domain of μ need not contain all the Borel sets. However, by an analog of Lemma <10> outlined in Problem [1], it could be extended to a Borel measure without disturbing the representation. □

<18> **Example.** Let $\mathcal{H}^+$ be a lattice cone of bounded continuous functions on a topological space, and let $T : \mathcal{H}^+ \to \mathbb{R}^+$ be a linear functional (necessarily increasing) with the property that to each $\epsilon > 0$ there exists a compact set K_ϵ for which $Th \le \epsilon$ if $0 \le h \le K_\epsilon^c$. (In Section 7.5, such a functional was called functionally tight.)

Suppose $1 \in \mathcal{H}^+$. The functional is automatically σ-smooth: if $1 \ge h_i \downarrow 0$ then eventually $K_\epsilon \subseteq \{h_i < \epsilon\}$, in which case $Th_i \le T\left((h_i - \epsilon)^+ + \epsilon\right) \le \epsilon + \epsilon T(1)$. In fact, the same argument shows that the functional is also τ-smooth, in the sense of Problem [2].

The functional T is represented by a measure μ on the sigma-field generated by $\mathcal{H}^+$. Suppose there exists a sequence $\{h_i\} \subseteq \mathcal{H}^+$ for which $1 \ge h_i \downarrow K_\epsilon$. (The version of the representation theorem for τ-smooth functionals, as described by Problem [2], shows that it is even enough to have $\mathcal{H}^+$ generate the underlying topology.) Then $\mu K_\epsilon = \lim_i Th_i = T(1) - \lim_i T(1 - h_i) \ge T(1) - \epsilon$. That is, μ is a tight measure, in the sense that it concentrates most of its mass on a compact set. It is inner regular with respect to approximation by the paving of compact sets. □

7. Problems

[1] A family of sets $\mathcal{U}$ is said to be downward filtering if to each pair U_1, U_2 in $\mathcal{U}$ there exists a U_3 in $\mathcal{U}$ with $U_1 \cap U_2 \supseteq U_3$. A set function $\mu : \mathcal{K}_0 \to \mathbb{R}^+$ is said to be ***τ-smooth*** if $\inf\{\mu K : K \in \mathcal{U}\} = \mu(\cap\mathcal{U})$ for every downward filtering family $\mathcal{U} \subseteq \mathcal{K}_0$. Write $\mathcal{K}$ for the $\cap a$-closure of a $(\emptyset, \cup f, \cap f)$ paving $\mathcal{K}_0$, the collection of all possible intersections of subclasses of $\mathcal{K}_0$.

(i) Show that $\mathcal{K}$ is a $(\emptyset, \cup f, \cap a)$ paving (stable under arbitrary intersections).

(ii) Show that a $\mathcal{K}_0$-tight set function that is τ-smooth at $\emptyset$ has a $\mathcal{K}$-tight, τ-additive extension to $\mathcal{K}$.

[2] Say that an increasing functional T on $\mathcal{H}^+$ is ***τ-smooth at zero*** if $\inf\{Th : h \in \mathcal{V}\}$ for each subfamily $\mathcal{V}$ of $\mathcal{H}^+$ that is downward filtering to the zero function. (That is, to each h_1 and h_2 in $\mathcal{V}$ there is an h_3 in $\mathcal{V}$ with $h_1 \wedge h_2 \geq h_3$ and the pointwise infimum of all functions in $\mathcal{V}$ is everywhere zero.) Extend Theorem <16> to τ-smooth functionals by constructing a $\mathcal{K}$-regular representing measure from the class $\mathcal{K}$ of sets representable as pointwise infima of subclasses of $\mathcal{H}^+$.

8. Notes

The construction via $\mathcal{K}$-tight inner measures is a reworking of ideas from Topsøe (1970). The application to integral representations is a special case of results proved by Pollard & Topsøe (1975).

The book by Fremlin (1974) contains an extensive treatment of the relationship between measures and linear functionals. The book by König (1997) develops the theory of measure and integration with a heavy emphasis on inner regularity.

See Pfanzagl & Pierlo (1969) for an exposition of the properties of pavings compact in the sense of Marczewski.

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