The extended real line

Homework Problem 1.5 was an inelegant attempt to convince you that $\mathcal{B}[0,\infty]$ is quite similar to $\mathcal{B}[0,\infty)$, with only a few minor changes involving ∞. Let me start again.

For the purpose of limit operations it is convenient to compactify the real line, by the addition of two new points: $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$. The ordering of $\mathbb{R}$ is extended to $\overline{\mathbb{R}}$ by specifying that $-\infty < r < +\infty$ for all $r \in \mathbb{R}$. Every subset of $\overline{\mathbb{R}}$ has a supremum and an infimium in $\overline{\mathbb{R}}$.

The topology of $\mathbb{R}$ is extended to $\overline{\mathbb{R}}$ by including sets $(r, +\infty]$ as neighborhoods of $+\infty$ and sets $[-\infty, r)$ as neighborhoods of $-\infty$, with $r \in \mathbb{R}$. A subset G of $\overline{\mathbb{R}}$ belongs to the set $\mathcal{G}_\infty$ of open sets (the topology for $\overline{\mathbb{R}}$) if and only if: for each x in G there is a neighborhood U of x for which $x \in U \subseteq G$. This definition ensures that the topology on $\mathbb{R}$ (that is, the set $\mathcal{G}$ of all open subsets of $\mathbb{R}$), is given by

<1> $$\mathcal{G} = \{G \cap \mathbb{R} : G \in \mathcal{G}_\infty\}.$$

A similar relationship exists between the σ-fields $\mathcal{B}(\mathbb{R}) := \sigma(\mathcal{G})$ and $\mathcal{B}(\overline{\mathbb{R}}) := \sigma(\mathcal{G}_\infty)$, namely

<2> $$\mathcal{B}(\mathbb{R}) = \{B \cap \mathbb{R} : B \in \mathcal{B}(\overline{\mathbb{R}})\}.$$

A generating class argument shows why.

> **Remark.** The following argument is tricky only because we have to keep track of whether a set B is considered a subset of $\mathbb{R}$ or of $\overline{\mathbb{R}}$.

First note the set—call it $\mathcal{B}_0$ for the moment—on the right-hand side of <2> is a σ-field of subsets of $\mathbb{R}$:

(i) $\emptyset = \emptyset \cap \mathbb{R}$

(ii) If $D = B \cap \mathbb{R} \in \mathcal{B}_0$ then $\mathbb{R}\backslash D = \mathbb{R} \cap (\overline{\mathbb{R}}\backslash B \in \mathcal{B}_0$. (Notice the way of avoiding the ambiguity in the symbol D^c: which might mean $\mathbb{R}\backslash D$ or $\overline{\mathbb{R}}\backslash D$.)

(iii) For a sequence $D_i = \mathbb{R} \cap B_i$ in $\mathcal{B}_0$ note that $\cup_i D_i = \mathbb{R} \cap (\cup_i B_i) \in \mathcal{B}_0$.

By equality <1>, $\mathcal{G} \subset \mathcal{B}_0$. It follows that $\mathcal{B}(\mathbb{R}) \subseteq \mathcal{B}_0$.

For the reverse inclusion consider

$$\mathcal{B}_1 = \{B \in \mathcal{B}(\overline{\mathbb{R}}) : B \cap \mathbb{R} \in \mathcal{B}(\mathbb{R})\}.$$

Arguing in a way similar to the previous paragraph, you can show that $\mathcal{B}_1$ is a σ-field on $\overline{\mathbb{R}}$. And $\mathcal{B}_1 \supseteq \mathcal{G}_\infty$ by equality <2>. It follows that $\mathcal{B}_1 = \mathcal{B}(\overline{\mathbb{R}})$. That is, $B \cap \mathbb{R} \in \mathcal{B}(\mathbb{R})$ for all $B \in \mathcal{B}(\overline{\mathbb{R}})$, which is another way of saying that $\mathcal{B}_0 \subseteq \mathcal{B}(\mathbb{R})$.

Finally, note that both $\{+\infty\}$ and $\{-\infty\}$ are both closed subsets of $\overline{\mathbb{R}}$, and hence they both belong to $\mathcal{B}(\overline{\mathbb{R}})$. If $B \in \mathcal{B}(\overline{\mathbb{R}})$ then all the sets

$$B\backslash\{+\infty\}, \quad B\backslash\{-\infty\}, \quad B\backslash\{+\infty, -\infty\} = B \cap \mathbb{R}$$

are also in $\mathbb{B}(\overline{\mathbb{R}})$. This gives a characterization of $\mathcal{B}(\overline{\mathbb{R}})$ similar to the one for $\mathcal{B}[0,\infty]$ on the homework.