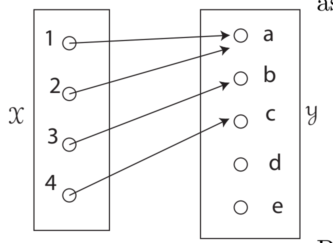
Statistics 330b/600b, Math 330b spring 2014
Homework # 1
Due: Thursday 23 January

Please attempt at least the starred problems.

*[1] Suppose T maps a set $\mathcal{X}$ into a set $\mathcal{Y}$. For each $B \subseteq \mathcal{Y}$ and $A \subseteq \mathcal{X}$ define $T^{-1}B := \{x \in \mathcal{X} : T(x) \in B\}$. and $T(A) := \{T(x) : x \in A\}$. Some of the following eight assertions are true in general and some are false.



$$
\begin{array}{llll}
(i) & T\left(\cup_i A_i\right) = \cup_i T(A_i) & (ii) & T^{-1}\left(\cup_i B_i\right) = \cup_i T^{-1}(B_i) \\
(iii) & T\left(\cap_i A_i\right) = \cap_i T(A_i) & (iv) & T^{-1}\left(\cap_i B_i\right) = \cap_i T^{-1}(B_i) \\
(v) & T\left(A^c\right) = \left(T\left(A\right)\right)^c & (vi) & T^{-1}\left(B^c\right) = \left(T^{-1}\left(B\right)\right)^c \\
(vii) & T^{-1}\left(T(A)\right) = A & (viii) & T\left(T^{-1}(B)\right) = B
\end{array}
$$

Provide counterexamples for each of the false assertions. You might find it helpful to contemplate the picture for a special case.

*[2] Suppose x and y are two points in a set $\mathcal{X}$ and $\mathcal{E}$ is a set of subsets of $\mathcal{X}$ with the property that $\mathbf{1}_E\{x\} = \mathbf{1}_E\{y\}$ for all $E \in \mathcal{E}$. Show that $\mathbf{1}_B\{x\} = \mathbf{1}_B\{y\}$ for all $B \in \sigma(\mathcal{E})$. Hint: Show that $\mathcal{B}_0 = \{B \in \sigma(\mathcal{E}) : \mathbf{1}_B\{x\} = \mathbf{1}_B\{y\}\}$ is a sigma-field.

*[3] Let $\mathcal{G}$ denote the set of all open subsets of $\mathbb{R}^2$ and $\mathcal{T}$ denote the set of all closed triangles in $\mathbb{R}^2$. Show that $\sigma(\mathcal{G}) = \sigma(\mathcal{T})$.

[4] Let $A_1, \dots, A_N$ be events in a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. For each subset J of $\{1, 2, \dots, N\}$ write A_J for $\cap_{i \in J} A_i$. Define $S_k := \sum_{|J|=k} \mathbb{P}A_J$, where $|J|$ denotes the number of indices in J.

$$\mathbb{P}\{\text{exactly } m \text{ of the } A_i\text{'s occur}\} = \binom{m}{m} S_m - \binom{m+1}{m} S_{m+1} + \dots \pm \binom{N}{m} S_N$$

Hint: Write $\mathbf{1}_i(\omega)$ for the indicator function of the set A_i and $\mathbf{1}_J(\omega)$ for the indicator function of A_J. For a dummy variable z, show that

$$\prod\nolimits_{i=1}^{N} (1 - \mathbf{1}_i(\omega) + z\mathbf{1}_i(\omega)) = \sum\nolimits_{k=0}^{n} (z-1)^k \sum\nolimits_{|J|=k} \mathbf{1}_J(\omega) \qquad \text{for every } \omega \in \Omega.$$

Expand the left-hand side, take expectations, then interpret the coefficient of z^m.

[5] Write $\mathcal{X}$ for $[0, +\infty]$. Construct a metric on $\mathcal{X}$ that corresponds to the usual notion of convergence (What is that?) for which the corresponding Borel sigma-field $\mathcal{B}(\mathcal{X})$ consists of all sets of the form $B \cap \mathcal{X}$ or $(B \cap \mathcal{X}) \cup \{+\infty\}$, with $B \in \mathcal{B}(\mathbb{R})$. (Explanations needed.) Hint: Consider $\psi(|x - y|)$ where ψ is a concave, increasing function from $[0, \infty]$ onto $[0, 1]$.