Statistics 330b/600b, Math 330b spring 2014
Homework # 10
Due: Thursday 10 April

*[1] Suppose $X \in \mathcal{L}^1(\Omega, \mathcal{F}, \mathbb{P})$ and $\{\mathcal{F}_n : n \in \mathbb{N}_0\}$ is a filtration: $\mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq \cdots \subseteq \mathcal{F}$. Define $\mathcal{E} = \cup_{n\in\mathbb{N}_0}\mathcal{F}_n$ and $\mathcal{F}_\infty = \sigma(\mathcal{E})$.

For each n in $\overline{\mathbb{N}}_0 = \mathbb{N}_0 \cup \{\infty\}$ define $X_n = \mathbb{P}_{\mathcal{F}_n}X$. In class I (almost) showed that $\{(X_n, \mathcal{F}_n) : n \in \overline{\mathbb{N}}_0\}$ is a martingale. I then deduced (via the positive supermartingale convergence theorem) that $\{X_n : n \in \mathbb{N}_0\}$ converges almost surely to the $\mathcal{F}_\infty$-measurable random variable $Z = \liminf_{\mathbb{N}_0} X_n$. [Be careful here: X_∞ is not involved in the liminf, which is $\lim_{n\in\mathbb{N}_0} \inf_{n\le i<\infty} X_i$.]

Without loss of generality assume $X \ge 0$. Show that $\{\mathbb{P}|X_n - Z| : n \in \mathbb{N}_0\}$ converges to zero and $Z = X_\infty$ almost surely by the following steps.

(i) Show $\mathbb{P}X \ge \mathbb{P}Z$.

(ii) For each $c \in \mathbb{R}^+$ and $n \in \mathbb{N}_0$ define $X_n(c) = \mathbb{P}_{\mathcal{F}_n}(X \wedge c)$. Show that there exists a nonnegative, $\mathcal{F}_\infty$-measurable random variable $Z(c)$ for which $\{X_n : n \in \mathbb{N}_0\}$ converges to $Z(c)$ almost surely and in $\mathcal{L}^1$ norm. Deduce that

$$\mathbb{P}Z \ge \sup_c \mathbb{P}(X \wedge c) = \mathbb{P}X.$$

(iii) Show that $\mathbb{P}|X_n - Z| = 2\mathbb{P}(Z - X_n)^+ \to 0$.

(iv) Show that $\mathbb{P}XE = \mathbb{P}ZE$ for all E in $\mathcal{E}$. Then complete the proof that $Z = X_\infty$ almost surely.

*[2] Suppose $X_n \to X$ almost surely, with $|X_n| \le H$ for an H in $\mathcal{L}^1(\Omega, \mathcal{F}, \mathbb{P})$. Let $\mathcal{A}$ be a sub-sigma-field of $\mathcal{F}$. Using only the properties for conditional expectations dereived in class, show that $\mathbb{P}_\mathcal{A} X_n \to \mathbb{P}_\mathcal{A} X$ almost surely.

*[3] Suppose $\{X_n\}$ is a sequence in $\mathcal{L}^2(\Omega, \mathcal{F}, \mathbb{P})$ that converges in $\mathcal{L}^2$ to some random variable X. Let $\mathcal{A}$ be a sub-sigma-field of $\mathcal{F}$. Show that $\mathbb{P}_\mathcal{A} X_n$ converges in $\mathcal{L}^2$ to $\mathbb{P}_\mathcal{A} X$.

[4] Suppose $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space and $T : (\Omega, \mathcal{F}) \to (\mathcal{T}, \mathcal{B})$ where $\mathcal{T}$ is a separable metric space equipped with its Borel sigma-field. Define $\mathbb{Q} = T\mathbb{P}$. Abbreviate $\mathcal{M}^+(\Omega, \mathcal{F})$ to $\mathcal{M}^+(\mathcal{F})$, and so on.

Suppose $\Lambda = \{\lambda_t : t \in \mathcal{T}\}$ is a set of probability measures on $\mathcal{F}$ for which $t \mapsto \lambda_t f$ is $\mathcal{B}$-measurable for each f in $\mathcal{M}^+(\mathcal{F})$. Suppose also that

$$\mathbb{P}g(T\omega)f(\omega) = \mathbb{Q}^t g(t)\lambda_t^\omega f(\omega) \qquad \text{for all } g \in \mathcal{M}^+(\mathcal{B}) \text{ and } f \in \mathcal{M}^+(\mathcal{F}).$$

(i) Use a generating class argument to show that $\mathbb{P}h(\omega, T\omega) = \mathbb{Q}^t\lambda_t^\omega h(\omega, t)$ for all $h \in \mathcal{M}^+(\mathcal{F} \otimes \mathcal{B})$.

(ii) Show that the function $h(\omega, t) = \{t \ne T\omega\}$ is $\mathcal{F} \otimes \mathcal{B}$-measurable.

(iii) Deduce that $\lambda_t\{\omega : t \ne T\omega\} = 0$ for $\mathbb{Q}$ almost all t.

[5] Suppose $X_0 \in \mathcal{L}^2(\Omega, \mathcal{F}, \mathbb{P})$. Suppose also that $\mathcal{A}_1$ and $\mathcal{A}_2$ are sub-σ-fields of $\mathcal{F}$. Define $\mathcal{A} = \mathcal{A}_1 \cap \mathcal{A}_2$. (Do not assume that $\mathcal{A}$ is equal to an $\mathcal{A}_i$. That is, the $\mathcal{A}_i$'s are not nested.) Define sequences of $\mathcal{A}_1$-measurable random variables $\{Y_n : n \in \mathbb{N}_0\}$ and $\mathcal{A}_2$-measurable random variables $\{X_n : n \in \mathbb{N}\}$ recursively, by $Y_n = \mathbb{P}_{\mathcal{A}_1} X_n$ and $X_{n+1} = \mathbb{P}_{\mathcal{A}_2} Y_n$. Suppose there exists a Z in $\mathcal{L}^2(\Omega, \mathcal{F}, \mathbb{P})$ for which $\mathbb{P}|X_n - Z|^2 \to 0$. Show that $Z = \mathbb{P}_\mathcal{A} X_0$ almost surely. [I would also like to know when the X_n's converge in $\mathcal{L}^2$, but that seems a bit hard without using some sort of compactness property. Compare with Breiman and Friedman 1985.]

References

Breiman, L. and J. H. Friedman (1985). Estimating optimal transformations for multiple regression and correlation. *Journal of the American Statistical Association 80*(391), 580–598.