Statistics 330b/600b, Math 330b spring 2014
Homework # 11
Due: Thursday 24 April

*[1] Suppose g_1 and g_2 are $\mathcal{A}\backslash\mathcal{B}$-measurable maps from $(\mathcal{X},\mathcal{A})$ to $(\mathcal{Y},\mathcal{B})$, with product-measurable graphs. Define $\psi_i(x) = (x, g_i(x))$, a measurable map from $\mathcal{X}$ into $\mathcal{X}\times\mathcal{Y}$. Let $P_i = \psi_i(\mu_i)$, for probability measures μ_i on $\mathcal{A}$. Let $P = \alpha_1 P_1 + \alpha_2 P_2$, for constants $\alpha_i > 0$ with $\alpha_1 + \alpha_2 = 1$. Let X denote the coordinate map from $\mathcal{X}\times\mathcal{Y}$ onto the $\mathcal{X}$ space and Q denote the distribution of X under P.

(i) For $f \in \mathcal{M}^+(\mathcal{X}\times\mathcal{Y}, \mathcal{A}\otimes\mathcal{B})$ find an expression for Pf in terms of μ_1, μ_2, g_1, and g_2.

(ii) Find an expression for Q in terms of μ_1 and μ_2.

(iii) Show that the set $\{x : g_1(x) \neq g_2(x)\}$ is $\mathcal{A}$-measurable if $\mathcal{Y}$ is a separable metric space and $\mathcal{B}$ is its Borel sigma-field,

(iv) Suppose the conditional probability distribution $P_x = P(\cdot \mid X = x)$ puts mass $\beta_i(x)$ at $(x, g_i(x))$, for $i = 1, 2$. Show that, on the set $\{g_1 \neq g_2\}$, the functions β_i are almost surely uniquely determined as Radon-Nikodym derivatives.

(v) Consider the special case where: $\mathcal{X} = \mathcal{Y} = \mathbb{R}$ and $\mathcal{A} = \mathcal{B} = \mathcal{B}(\mathbb{R})$; the measure μ_i is the $N(\theta_i, \sigma_i^2)$; and $g_i(x) = a_i + b_i x$ for constants a_i and b_i. Find the conditional distribution $P(\cdot \mid X = x)$.

*[2] Suppose $\mathcal{X}$ is a set equipped with a metric d. Show that $\mathrm{BL}(\mathcal{X})$ is a vector space that is stable under pairwise products and pairwise maxima. [For maxima, it is easier to first show that $g^+ \in \mathrm{BL}(\mathcal{X})$ if $g \in \mathrm{BL}(\mathcal{X})$.]

*[3] Suppose X_n has a $N(\mu_n, \sigma_n^2)$ distribution, and $X_n \rightsquigarrow P$, for some probability measure P on $\mathcal{B}(\mathbb{R})$.

(i) Explain why there exists a finite constant M such that $\liminf \mathbb{P}\{|X_n| \le M\} > 3/4$.

(ii) Explain why we must have $|\mu_n| \le M$ and $\sigma_n \le 8M/(3\sqrt{2\pi})$ eventually.

(iii) Show that $\mu := \lim \mu_n$ and $\sigma^2 := \lim \sigma_n^2$ must exist as finite limits and that P must be the $N(\mu, \sigma^2)$ distribution.

(iv) Extend the result to sequences of random vectors.

[4] Let $\{X_{n,i}\}$ be a triangular array of random variables, independent within each row and satisfying

(a) $\max_i |X_{n,i}| \to 0$ in probability,

(b) $\sum_i \mathbb{P}X_{n,i}\{|X_{n,i}| \le \epsilon\} \to \mu$ for each $\epsilon > 0$,

(c) $\sum_i \mathrm{var}\big(X_{n,i}\{|X_{n,i}| \le \epsilon\}\big) \to \sigma^2 < \infty$ for each $\epsilon > 0$.

(i) Show that there exists a sequence of positive numbers $\{\epsilon_n\}$ that tends to zero slowly enough that

(d) $\mathbb{P}\{\max_i |X_{n,i}| > \epsilon_n\} \to 0$,

(e) $\sum_i \mathbb{P}X_{n,i}\{|X_{n,i}| \le \epsilon_n\} \to \mu$,

(f) $\sum_i \mathrm{var}\big(X_{n,i}\{|X_{n,i}| \le \epsilon_n\}\big) \to \sigma^2$.

(ii) Deduce that $\sum_i X_{n,i} \rightsquigarrow N(\mu, \sigma^2)$. Hint: Consider $\eta_{n,i} := X_{n,i}\{|X_{n,i}| \le \epsilon_n\}$ and $\xi_{n,i} := \eta_{n,i} - \mathbb{P}\eta_{n,i}$.

[5] For each i in some (possibly uncountably infinite) index set $\mathcal{I}$, suppose $h_i : \mathcal{X} \to \mathcal{Y}_i$. Suppose each $\mathcal{Y}_i$ is equipped with a sigma-field $\mathcal{B}_i$. Equip the product space $\mathcal{Y} = \prod_{i\in\mathcal{I}} \mathcal{Y}_i$ with its product sigma-field $\mathcal{B}$, which is the sigma-field generated by the collection $\mathcal{E}$ of all product sets of the form $\prod_{i\in\mathcal{I}} B_i$ with $B_i \in \mathcal{B}_i$ for all i and $B_i = \mathcal{Y}_i$ for all except finitely many i.

(i) Show that there is a smallest sigma-field $\mathcal{A}$ on $\mathcal{X}$ that makes h_i an $\mathcal{A}\backslash\mathcal{B}_i$-measurable function for every i.

(ii) For each f in $\mathcal{M}^+(\mathcal{X}, \mathcal{A})$ show that there exists a g in $\mathcal{M}^+(\mathcal{Y}, \mathcal{B})$ for which $f = g \circ H$, where $H : \mathcal{X} \to \mathcal{Y}$ is defined as the map for which $H(x)_i = h_i(x)$.