Statistics 330b/600b, Math 330b spring 2014
Homework # 2
Due: Thursday 30 January

Please attempt at least the starred problems.

Throughout this sheet, $\mathcal{A}$ is a sigma-field on some set $\mathcal{X}$ and μ is a measure on $\mathcal{A}$ and $\mathcal{N}_\mu = \{N \in \mathcal{A} : \mu N = 0\}$.

*[1] Suppose $f_1, \ldots, f_k \in \mathcal{M}^+(\mathcal{X}, \mathcal{A})$ and $\theta_1, \ldots, \theta_k$ are strictly positive numbers that sum to one. Let μ be a measure on $\mathcal{A}$. Show that

$$\mu \prod\nolimits_{i \le k} f_i^{\theta_i} \le \prod\nolimits_{i \le k} (\mu f_i)^{\theta_i}$$

by following these steps.

(i) Explain why the inequality is trivially true if $\mu f_i = 0$ for at least one i or if $\mu f_i = \infty$ for at least one i (and all the other μf_j are strictly positive).

(ii) Explain why there is no loss of generality in assuming that $\mu f_i = 1$ for each i and $f_i(x) < \infty$ for each x and i.

(iii) For all $a_1, \ldots, a_k \in \mathbb{R}^+$, show that $\prod_{i \le k} a_i^{\theta_i} \le \sum_{i \le k} \theta_i a_i$. Hint: First dispose of the trivial case where at least one a_i is zero, then rewrite the inequality using $b_i = \log a_i$. *You do not need to reprove that the* log *function is concave on* $(0, \infty)$.

(iv) Complete the proof by considering the inequality from (iii) with $a_i = f_i(x)$.

> **Remark.** Textbooks often contain the the special case where $k = 2$ and $\theta_1 = 1/p$ and $\theta_2 = 1/q$ and $f_1 = |g_1|^p$ and $f_2 = |g_2|^q$, with the assertion that $|\mu(g_1 g_2)| \le \mu|g_1 g_2| \le (\mu|g_1|^p)^{1/p} (\mu|g_2|^q)^{1/q}$.

*[2] For f in $\mathcal{L}^1(\mu)$ define $\|f\|_1 = \mu|f|$. Let $\{f_n\}$ be a Cauchy sequence in $\mathcal{L}^1(\mu)$, that is, $\|f_n - f_m\|_1 \to 0$ as $\min(m, n) \to \infty$. Show that there exists an f in $\mathcal{L}^1(\mu)$ for which $\|f_n - f\|_1 \to 0$, by following these steps. Note: Don't confuse Cauchy sequences (in $\mathcal{L}^1$ distance) of functions with Cauchy sequences of real numbers.

(i) For each $k \in \mathbb{N}$ there exists an $n(k) \in \mathbb{N}$ for which: $\|f_n - f_m\|_1 < 2^{-k}$ when $\min(m, n) \ge n(k)$. Without loss of generality assume that $n(k)$ is strictly increasing with k. Define $H(x) := \sum_{k=1}^{\infty} |f_{n(k)}(x) - f_{n(k+1)}(x)|$. Show that $\mu H < \infty$.

(ii) Show that there exists a real-valued, measurable function f for which

$$H \ge |f_{n(k)}(x) - f(x)| \to 0 \qquad \text{as } k \to \infty, \text{ for each } x \text{ with } H(x) < \infty.$$

Hint: $\mathbb{R}$ is complete. Be careful how you define $f(x)$ when $H(x) = \infty$.

(iii) Deduce that $\left\|f_{n(k)} - f\right\|_1 \to 0$ as $k \to \infty$.

(iv) Show that f belongs to $\mathcal{L}^1(\mu)$ and $\|f_n - f\|_1 \to 0$ as $n \to \infty$.

[3] Suppose $\{f_n : n \in \mathbb{N}\} \subset \mathcal{L}^1(\mathcal{X}, \mathcal{A}, \mu)$ and $\sup_n |f_n(x)| \le F(x)$ for each x, for some $F \in \mathcal{L}^1(\mathcal{X}, \mathcal{A}, \mu)$. Suppose also that $\lim_n \mu\{x : |f_n(x)| > \epsilon F(x)\} = 0$ for each $\epsilon > 0$. Show that $\mu|f_n| \to 0$. Hint: If g is a real function for which $\sup_x |g(x)| \le 1$ then $|g| \le \epsilon + \{|g| > \epsilon\}$. Proof?

[4] Define $\overline{\mathcal{M}}^+ = \overline{\mathcal{M}}^+(\mathcal{X}, \mathcal{A}, \mu)$ to consist of all those functions f mapping $\mathcal{X}$ into $[0, \infty]$ for which there exists $g, h \in \mathcal{M}^+ = \mathcal{M}^+(\mathcal{X}, \mathcal{A}, \mu)$ with $g(x) \le f(x) \le g(x) + h(x)$ for all x and $\mu h = 0$. You should NOT assume that $f \in \mathcal{M}^+$. Call the pair $g, g+h$ an $\mathcal{M}^+$-sandwich for f.

(i) Show that there is no ambiguity in defining $\overline{\mu} : \overline{\mathcal{M}}^+ \to [0, \infty]$ by $\overline{\mu} f = \mu g$ for an arbitrarily chosen $\mathcal{M}^+$-sandwich for f. That is, show that if $g_1, g_1 + h_1$ and $g_2, g_2 + h_2$ are both sandwiches for f then $\mu g_1 = \mu g_2$.

(ii) Define $\overline{\mathcal{A}} := \{D \subseteq \mathcal{X} : \mathbf{1}_D \in \overline{\mathcal{M}}^+\}$. Show that $\overline{\mathcal{A}}$ is a sigma-field with $\overline{\mathcal{A}} \supseteq \mathcal{A}$. Show also that if E is a subset of $\mathcal{X}$ for which $E \subseteq N$, for some $N \in \mathcal{N}_\mu$, then $E \in \overline{\mathcal{A}}$.

(iii) Show that $\overline{\mathcal{M}}^+ = \mathcal{M}(\mathcal{X}, \overline{\mathcal{A}})$.

(iv) Show that $\overline{\mu}$ defines an increasing, linear functional on $\overline{\mathcal{M}}^+$ with the Monotone Convergence property.

(v) Show that the restriction of $\overline{\mu}$ to (the indicator functions of sets in) $\overline{\mathcal{A}}$ is a measure and that $\overline{\mu}$ is the integral with respect to that measure.