Statistics 330b/600b, Math 330b spring 2014
Homework # 3
Due: Thursday 6 February

Please attempt at least the starred problems.

[1] Let Ψ be a convex, increasing function for which $\Psi(0) = 0$ and $\Psi(x) \to \infty$ as $x \to \infty$. In class I defined $\mathcal{L}^\Psi(\mathcal{X}, \mathcal{A}, \mu)$ to be the set of all real-valued measurable functions f on $\mathcal{X}$ for which $\mu\Psi(|f|/c_0) < \infty$ for at least one positive real constant c_0. I also defined $\|f\|_\Psi := \inf\{c > 0 : \mu\Psi(|f|/c) \le 1\}$ for $f \in \mathcal{L}^\Psi(\mathcal{X}, \mathcal{A}, \mu)$. Prove (rigorously) that the infimum is achieved provided $0 < \|f\|_\Psi$.

*[2] A set $\mathcal{H}$ of bounded, real-valued functions on a set $\mathcal{X}$ is called a λ-space if:

(a) It is a vector space under the operations of pointwise addition and pointwise multiplication by constants.

(b) The constant function $\mathbf{1}$ belongs to $\mathcal{H}$.

(c) If $\{h_n : n \in \mathbb{N}\} \subset \mathcal{H}$ and $h_n(x) \uparrow h(x)$ for each x and $\sup_x h(x) < \infty$ then $h \in \mathcal{H}$.

Define $\mathcal{A} := \{A \subseteq \mathcal{X} : \mathbf{1}_A \in \mathcal{H}\}$. Initially suppose also that $\mathcal{H}$ is stable under pairwise products of functions. Follow the first six steps to show that $\mathcal{A}$ is a sigma-field and $\mathcal{H}$ consists of precisely those bounded real functions that are $\mathcal{A}$-measurable.

(i) If $\{h_n : n \in \mathbb{N}\} \subset \mathcal{H}$ and $\sup_x |h_n(x) - h(x)| \to 0$, show that $h \in \mathcal{H}$. Hint: Take a subsequence $\{n(k) : k \in \mathbb{N}\}$ along which $\sup_x |h_{n(k)}(x) - h(x)| \le \delta_k := 2^{-k}$. Then show that

$$h_{n(k+1)}(x) + \delta_{k+1} + \delta_k \ge h(x) + \delta_k \ge h_{n(k)}(x).$$

Deduce that $h_{n(k)} + R_k$, for $R_k = 3\sum_{i\le k} \delta_i$, increases for to $h(x) + 3$, for each x.

(ii) Suppose $h \in \mathcal{H}$ and $\sup_x |h(x)| = M < \infty$. By a small variation on Problem [3], there exists polynomials p_n for which $\sup_{|t|\le M} |p_n(t) - |t|| \to 0$. Deduce that $h_n := p_n(h)$ converges uniformly to $|h|$, so that $|h| \in \mathcal{H}$.

(iii) If $h_1, h_2 \in \mathcal{H}$ show that $h_1 \vee h_2$ and $h_1 \wedge h_2$ both belong to $\mathcal{H}$. Hint: $a \vee b = (a + b + |a - b|)/2$ for all $a, b \in \mathbb{R}$.

(iv) Show that $\mathcal{A}$ is a sigma-field. Hint: $\cup_{n\in\mathbb{N}} A_n = \lim_n \max_{i\le n} A_i$.

(v) If $h \in \mathcal{H}$ define $h_1 := \min(1, h^+)$. Show that $h_1 \in \mathcal{H}$ and that $\{h \ge 1\} = \lim_{n\to\infty} h_1^n$ belongs to $\mathcal{H}$. Deduce that h is $\mathcal{A}$-measurable. *Note: It is not enough just to have* $\{h \ge 1\} \in \mathcal{A}$.

(vi) Show that each bounded, $\mathcal{A}$-measurable function belongs to $\mathcal{H}$. Hint: Increasing limits of simple functions.

Now suppose $\mathcal{G}$ is a set of bounded real functions on $\mathcal{X}$ and $\mathcal{H}$ is the smallest λ-space for which $\mathcal{H} \supseteq \mathcal{G}$. Suppose also that $\mathcal{G}$ is stable under pairwise products.

(vii) Show that $\mathcal{H}_1 := \{h_1 \in \mathcal{H} : h_1 g \in \mathcal{H} \text{ for all } g \in \mathcal{G}\}$ is a λ-space. Deduce that $\mathcal{H}_1 = \mathcal{H}$.

(viii) Show that $\mathcal{H}_2 := \{h_2 \in \mathcal{H} : h_2 h \in \mathcal{H} \text{ for all } h \in \mathcal{H}\}$ is a λ-space. Deduce that $\mathcal{H}_2 = \mathcal{H}$, that is, $\mathcal{H}$ is stable under pairwise products.

(ix) Write $\sigma(\mathcal{G})$ for the smallest sigma-field on $\mathcal{X}$ for which each $g \in \mathcal{G}$ is $\sigma(\mathcal{G})\backslash\mathcal{B}(\mathbb{R})$-measurable. Show that $\sigma(\mathcal{G}) \subseteq \sigma(\mathcal{H}) = \mathcal{A}$. Deduce that every bounded, $\sigma(\mathcal{G})$-measurable real function belongs to $\mathcal{H}$.

*[3] For each θ in $[0,1]$ let X_θ have a Binomial(n,θ) distribution. That is,

$$\mathbb{P}\{X_\theta = k\} = \binom{n}{k}\theta^k(1-\theta)^{n-k} \qquad \text{for } k = 0,1,\ldots,n.$$

You may assume without proof that $\mathbb{P}X_\theta = n\theta$ and $\mathbb{P}(X_\theta - n\theta)^2 = n\theta(1-\theta)$.

Let f be a continuous function defined on $[0,1]$. Remember that f must also be uniformly continuous: for each fixed $\epsilon > 0$ there exists a $\delta_\epsilon > 0$ such that

$$(*) \qquad |f(s) - f(t)| \le \epsilon \qquad \text{whenever } |s-t| \le \delta_\epsilon, \text{ for } s,\ t \text{ in } [0,1].$$

Remember also that $|f|$ must be uniformly bounded, say, $\sup_x |f(x)| = M < \infty$.

(i) Show that $p_n(\theta) := \mathbb{P}f(X_\theta/n)$ is a polynomial in θ.

(ii) Show that $|f(X_\theta/n) - f(\theta)| \le \epsilon + 2M|X_\theta - n\theta|^2/(n\delta_\epsilon)^2$.

(iii) Deduce that $\sup_{0\le\theta\le1} |p_n(\theta) - f(\theta)| < 2\epsilon$ for n large enough. That is, deduce that $f(\cdot)$ can be uniformly approximated by polynomials over the range $[0,1]$, a result known as the ***Weierstrass approximation theorem***.