Statistics 330b/600b, Math 330b spring 2014
Homework # 4
Due: Thursday 13 February

Please attempt at least the starred problems.

*[1] Suppose $\mathcal{A}$ is a sigma-field on a set $\mathcal{X}$ and $\mathcal{B}$ is a countably generated sigma-field on a set $\mathcal{Y}$, that is, $\mathcal{B} = \sigma(\mathcal{E})$ for some countable $\mathcal{E} \subseteq \mathcal{B}$. Suppose also that $\mathcal{B}$ separates the points of $\mathcal{Y}$: if $y_1 \neq y_2$ then there exists a set $B \in \mathcal{B}$ for which $y_1 \in B$ and $y_2 \in B^c$. Without loss of generality $\mathcal{E}$ is stable under the formation of complements, so that, by the result from HW1.2, $\mathcal{E}$ also separates the points of $\mathcal{Y}$.

Suppose T is an $\mathcal{A}\backslash\mathcal{B}$-measurable map from $\mathcal{X}$ into $\mathcal{Y}$. Define $\text{graph}(T) := \{(x, Tx) : x \in \mathcal{X}\}$.

(i) Define $H := \cup_{E\in\mathcal{E}}(T^{-1}(E^c)) \times E$. Show that $H \subseteq \text{graph}(T)^c$.

(ii) If $y \neq Tx$, with $x \in \mathcal{X}$ and $y \in \mathcal{Y}$, show that $(x, y) \in H$. Deduce that $\text{graph}(T)^c \subseteq H$ and hence $\text{graph}(T) \in \mathcal{A} \otimes \mathcal{B}$.

> **Remark.** A topology $\mathcal{G}$ (= all open subsets of $\mathcal{X}$) on a set $\mathcal{X}$ is said to be countably generated if there exists a countable subset $\mathcal{G}_0$ of $\mathcal{G}$ such that $G = \cup\{G_0 \in \mathcal{G}_0 : G_0 \subseteq G\}$ every each $G \in \mathcal{G}$. For example, the usual topologies on Euclidean spaces are countably generated: take $\mathcal{G}_0$ as the set of all open balls with rational radii and centers in some fixed countable dense subset.

[2] Let $(\mathcal{X}_1, \mathcal{G}_1)$ and $(\mathcal{X}_2, \mathcal{G}_2)$ be topological spaces equipped with their Borel sigma-fields $\mathcal{B}(\mathcal{X}_i) = \sigma(\mathcal{G}_i)$. Equip $\mathcal{X}_1 \times \mathcal{X}_2$ with the product topology and its Borel sigma-field $\mathcal{B}(\mathcal{X}_1 \times \mathcal{X}_2)$. (The open sets in the product space are, by definition, all possible unions of sets $G_1 \times G_2$, with G_i open in $\mathcal{X}_i$.)

If you are unfamiliar with general topology you may assume $\mathcal{X}_1 = \mathcal{X}_2 = \mathbb{R}$.

(i) Show that $\mathcal{B}(\mathcal{X}) \otimes \mathcal{B}(\mathcal{Y}) \subseteq \mathcal{B}(\mathcal{X} \times \mathcal{Y})$. Hint: First show that the set $\mathcal{B}_1 = \{B_1 \in \mathcal{B}(\mathcal{X}_1) : B_1 \times \mathcal{X}_2 \in \mathcal{B}(\mathcal{X}_1 \times \mathcal{X}_2)\}$ is a σ-field for which $\mathcal{B}_1 \supseteq \mathcal{G}_1$.

(ii) If both $\mathcal{X}$ and $\mathcal{Y}$ have countably generated topologies, prove $\mathcal{B}(\mathcal{X}) \otimes \mathcal{B}(\mathcal{Y}) = \mathcal{B}(\mathcal{X} \times \mathcal{Y})$.

*[3] Suppose X is a real-valued random variable, defined on a set Ω equipped with a sigma-field $\mathcal{F}$. Show that the set $\{(\omega, t) \in \Omega \times \mathbb{R} : X(\omega) > t\}$ belongs to $\mathcal{F} \otimes \mathcal{B}(\mathbb{R})$.

[4] Let $(\mathcal{X}, \mathcal{A}, \mu)$ and $(\mathcal{Y}, \mathcal{B}, \nu)$ be two measure spaces, with both μ and ν sigma-finite. Write $\mathcal{G}$ for the set of all functions expressible as finite linear combinations of measurable rectangles. That is, a typical g in $\mathcal{G}$ is expressible as a finite sum $\sum_{i=1}^{k} \alpha_i\{x \in A_i,\ y \in B_i\}$ for some sets $A_i \in \mathcal{A}$ and $B_i \in \mathcal{B}$ and real numbers α_i, for $i = 1, 2, \ldots, k$.

Show that for each f in $\mathcal{L}^2(\mathcal{X} \times \mathcal{Y}, \mathcal{A} \otimes \mathcal{B}, \mu \otimes \nu)$ and each $\epsilon > 0$ there exist a $g \in \mathcal{G}$ such that $\mu \otimes \nu|f - g|^2 < \epsilon^2$. Follow these steps.

(i) First suppose that both μ and ν are finite measures and $|f|$ is bounded. Use a lambda-space argument to establish the asserted approximation property.

(ii) Extend to the sigma-finite case with f possibly unbounded. Hint: First approximate the function f by some $f_n := (-n) \vee (f \wedge n)$.