Statistics 330b/600b, Math 330b spring 2014
Homework # 6
Due: Thursday 27 February

Please attempt at least the starred problems.

*[1] Suppose X and Y are independent real-valued random variables for which $X = Y$ almost surely. Use Fubini to show that there must exist some $c \in \mathbb{R}$ for which $X = c$ almost surely.

*[2] Suppose T is a function from a set $\mathcal{X}$ into a set $\mathcal{Y}$, which is equipped with a σ-field $\mathcal{B}$. Recall that $\sigma(T) := \{T^{-1}B : B \in \mathcal{B}\}$ is the smallest sigma-field on $\mathcal{X}$ for which T is $\sigma(T)\backslash\mathcal{B}$-measurable.

Show that to each f in $\mathcal{M}^+(\mathcal{X}, \sigma(T))$ there exists a g in $\mathcal{M}^+(\mathcal{Y}, \mathcal{B})$ such that $f = g \circ T$ (that is, $f(x) = g(T(x))$, for all x in $\mathcal{X}$) by following these steps.

(i) If f is the indicator function of $T^{-1}(B)$ and g is the indicator function of B, show that $f = g \circ T$.

(ii) Consider the case where $f \in \mathcal{M}^+_{\text{simple}}(\mathcal{X}, \sigma(T))$.

(iii) Suppose $f_n = g_n \circ T$ is a sequence in $\mathcal{M}^+_{\text{simple}}(\mathcal{X}, \sigma(T))$ that increases pointwise to f. Define $g(y) = \limsup g_n(y)$ for each y in $\mathcal{Y}$. Show that g has the desired property.

(iv) In part (iii), why can't you assume that $\lim g_n(y)$ exists for each y in $\mathcal{Y}$?

*[3] In class I sketched some of the ideas needed to prove the following version of the SLLN: if $X_1, X_2, \dots$ are independent and identically distributed random variables with $\mathbb{P}|X_1| < \infty$ then $(X_1 + \cdots + X_n)/n \to \mathbb{P}X_1$ almost surely. Use this result to deduce the following extension: Suppose $Y_1, Y_2, \dots$ are independent and identically distributed random variables taking values in $[0, \infty)$ for which $\mathbb{P}Y_1 = +\infty$. Then $(Y_1 + \cdots + Y_n)/n \to \infty$ almost surely.

[4] Suppose $X_1, X_2, \dots$ are independent random variables with $\mathbb{P}X_i = 0$ for each i and $\sup_i \mathbb{P}X_i^{2k} = M < \infty$ for some positive integer k. Provide a convincing argument in less than a page to show that

$$\mathbb{P}\left(X_1 + \cdots + X_n\right)^{2k} = O(n^k).$$