Statistics 330b/600b, Math 330b spring 2014
Homework # 7
Due: Thursday 6 March

Please attempt at least the starred problems.

*[1] Suppose $\xi \in \mathcal{L}^1(\Omega, \mathcal{F}, \mathbb{P})$ and $\mathcal{G}$ is a sub-σ-field of $\mathcal{F}$.

(i) Suppose $\mathbb{P}\xi G = 0$ for all $G \in \mathcal{G}$. Show that $\mathbb{P}(\xi g) = 0$ for all $\mathcal{G}$-measurable random variables g for which $\mathbb{P}|\xi g| < \infty$.

(ii) Suppose $\mathbb{P}\xi G \geq 0$ for all $G \in \mathcal{G}$. For which $\mathcal{G}$-measurable random variables g can you conclude that $\mathbb{P}(\xi g) \geq 0$? Proof required.

*[2] Suppose W and Z are nonnegative random variables for which, for some positive constants β and C,

$$t\mathbb{P}\{W > \beta t\} \leq C\mathbb{P}Z\{W > t\} \qquad \text{for all } t > 0.$$

Suppose also that $\mathbb{P}|Z|^p < \infty$ for some p in $(1, \infty)$. Show that

$$\|W\|_p \leq Cp\beta^p \|Z\|_p /(p-1).$$

To make things slightly easier, I will let you assume that $\|W\|_p < \infty$. For a truly virtuoso effort, you might also show that the finiteness of $\|W\|_p$ actually follows from the finiteness of $\|Z\|_p$.

[3] Suppose $\{(S_i, \mathcal{F}_i) : i = 0, 1, 2, \ldots, N\}$ is a nonnegative submartingale. For some fixed $p > 1$ suppose $\mathbb{P}S_i^p < \infty$ for each i. Let $q > 1$ be defined by $p^{-1} + q^{-1} = 1$. Define $M_n = \max_{i \leq n} S_i$. Show that

$$\mathbb{P}\left(M_n^p\right) \leq q^p \mathbb{P}S_n^p.$$

You may use the result from Problem [2] and the upper bound for $x\mathbb{P}\{M_n \geq x\}$ proved with the Stopping Time Lemma.

[4] Suppose $\{(X_n, \mathcal{F}_n) : n \in \mathbb{N}_0\}$ is a martingale for which $\sup_{n \in \mathbb{N}_0} \mathbb{P}X_n^2 < \infty$.

Write X_n as a sum of increments, $X_n = X_0 + \sum_{1 \leq i \leq n} \xi_i$. In class I showed that $\mathbb{P}X_n^2 = \mathbb{P}X_0^2 + \sum_{1 \leq i \leq n} \sigma_i^2$, where $\sigma_i^2 = \mathbb{P}\xi_i^2$. Thus $\sum_{i \in \mathbb{N}} \sigma_i^2 < \infty$.

(i) Show that $\{X_n\}$ is a Cauchy sequence in $\mathcal{L}^2(\Omega, \mathcal{F}, \mathbb{P})$ and hence $\mathbb{P}|X_n - Z|^2 \to 0$ as $n \to \infty$ for some Z in $\mathcal{L}^2(\Omega, \mathcal{F}, \mathbb{P})$.

(ii) For each $\epsilon > 0$, show that

$$\mathbb{P}\{\sup_{i > n} |X_i - X_n| > \epsilon\} \leq R(n)/\epsilon^2 := \epsilon^{-2} \sum_{k \geq n+1} \sigma_k^2.$$

(iii) Show that there exists an increasing sequence of integers $n(1) < n(2) < \ldots$ for which

$$\sum_{k \in \mathbb{N}} \mathbb{P}\{\sup_{i > n(k)} |X_i - X_{n(k)}| > k^{-1}\} < \infty.$$

(iv) Deduce that $\{X_n(\omega)\}$ is a Cauchy sequence of real numbers for $\mathbb{P}$ almost all ω.

(v) Deduce that $X_n \to Z$ almost surely.