Statistics 330b/600b, Math 330b spring 2014
Homework # 8
Due: Thursday 27 March

Please attempt at least the starred problems.

*[1] Suppose σ is a stopping times for a filtration $\{\mathcal{F}_i : i \in \mathbb{N}_0\}$ and $\{X_i : i \in \mathbb{N}_0\}$ is a sequence of real-valued random variables adapted to the same filtration. Suppose also that $\{B_i : i \in \mathbb{N}_0\} \subset \mathcal{B}(\mathbb{R})$. Show that $\tau = \inf\{i \geq \sigma : X_i \in B_i\}$ is also a stopping time.

*[2] Suppose σ and τ are stopping times for a filtration $\{\mathcal{F}_i : i \in \mathbb{N}_0\}$. Show that $\mathcal{F}_{\sigma\wedge\tau} = \mathcal{F}_\sigma \cap \mathcal{F}_\tau$. You might also try to prove the result with $\mathbb{N}_0$ replaced by $\mathbb{R}^+$.

*[3] (Projections in Hilbert space) Let $\mathcal{K}$ be a closed, convex subset of $\mathcal{H} := \mathcal{L}^2(\mathcal{X}, \mathcal{A}, \mu)$. For a fixed f in $\mathcal{H}\backslash\mathcal{K}$ define $\delta := \inf\{\|f - h\|_2 : h \in \mathcal{K}\}$. In this problem you will show that there is an f_0 (unique up to μ-equivalence) in $\mathcal{K}$ for which $\|f - f_0\|_2 = \delta$ and establish some related properties.

(i) For all $a, b \in \mathcal{H}$ show that

$$\|a + b\|_2^2 + \|a - b\|_2^2 = 2\,\|a\|_2^2 + 2\,\|b\|_2^2 .$$

(ii) If $h_n \in \mathcal{K}$ are chosen to make $\delta_n := \|f - h_n\|_2 \to \delta$, show that $\{h_n : n \in \mathbb{N}\}$ is a Cauchy sequence in $\mathcal{H}$. Assuming completeness of $\mathcal{H}$, deduce that there exists some f_0 in $\mathcal{K}$ for which $\|h_n - f_0\|_2 \to 0$. Explain why $\|f - f_0\|_2 = \delta$.

(iii) For all $h \in \mathcal{K}$ and all $t \in [0,1]$ explain why $\|f - (1-t)f_0 - th\|_2 \geq \delta$. Deduce that $\mathcal{K}$ is contained in the closed halfspace $\{h \in \mathcal{H} : \langle h - f_0, f_0 - f\rangle \geq 0\}$.

(iv) If $\mathcal{K}$ is actually a closed subspace of $\mathcal{H}$, deduce that $\langle h, f_0 - f\rangle = 0$ for all h in $\mathcal{K}$.

*[4] Suppose λ and μ are finite measures defined on a sigma-field $\mathcal{A}$ of subsets of a set $\mathcal{X}$. In class I will show that, if $\lambda A \leq \mu A$ for each $A \in \mathcal{A}$ then there exists an $\mathcal{A}$-measurable function Δ_0 for which $0 \leq \Delta_0 \leq 1$ and

$$\lambda f = \mu(f\Delta_0) \qquad \text{for all } f \text{ in } \mathcal{M}^+(\mathcal{X}, \mathcal{A}). \tag{$*$}$$

(In fact I have already establish this result by martingale methods for the special case where $\mathcal{A}$ is countably generated.)

(i) Suppose λ and ν are finite measures with $\lambda \ll \nu$. That is, for all $A \in \mathcal{A}$, if $\nu A = 0$ then $\lambda A = 0$. Apply $(*)$ with $\mu = \nu + \lambda$. Deduce that $\mu\Delta_0\{\Delta_0 \geq 1\} = 0$ then show that $\lambda f = \nu(f\Delta)$ for all $f \in \mathcal{M}^+(\mathcal{X}, \mathcal{A})$, where $\Delta := \Delta_0\{\Delta_0 < 1\}/(1 - \Delta_0)$ Beware of $\infty - \infty$.

(ii) Show that the Δ from part (i) is unique up to ν-equivalence.

(iii) Extend the result to the case where λ and ν are sigma-finite.

[5] (Hard) Suppose τ is a stopping time for the natural filtration $\{\mathcal{F}_i : i \in \mathbb{N}_0\}$ generated by real-valued random variables $\{X_i : i \in \mathbb{N}_0\}$ on a set Ω. (That is, $\mathcal{F}_i$ is the smallest sigma-field on Ω for which X_j is $\mathcal{F}_i\backslash\mathcal{B}(\mathbb{R})$-measurable for $0 \leq j \leq i$.) Define new random variables $Z_i = X_{\tau\wedge i}$, for $i \in \mathbb{N}_0$. Define $\mathcal{G}$ to be the smallest sigma-field on Ω for which each Z_i, for $i \in \mathbb{N}_0$, is $\mathcal{G}\backslash\mathcal{B}(\mathbb{R})$-measurable. Show that $\mathcal{G} = \mathcal{F}_\tau$.