Statistics 330b/600b, Math 330b spring 2014
Homework # 9
Due: Thursday 3 April

*[1] Suppose $\{X_n\}$ is a sequence of real-valued random variables for which: to each $\epsilon > 0$ and $\delta > 0$ there exists $n_0 = n_0(\epsilon, \delta)$ such that

$$\mathbb{P}\{|X_n - X_m| > \delta\} < \epsilon \qquad \text{for all } m, n \geq n_0.$$

Show that there exists a real-valued random variable X for which

$$\mathbb{P}\{|X_n - X| > \delta\} \to 0 \qquad \text{as } n \to \infty$$

by the folllowing steps.

(i) Show that there exists an increasing sequence of integers $n(k)$ for which

$$\sum\nolimits_{k\in\mathbb{N}} \mathbb{P}\{|X_{n(k)} - X_{n(k+1)}| > 2^{-k}\} < \infty$$

(ii) Deduce that there exists a real-valued random variable X for which $X_{n(k)} \to X$ almost surely as $k \to \infty$.

(iii) For each $\delta > 0$, deduce that $\mathbb{P}\{|X_{n(k)} - X| > \delta\} \to 0$ as $k \to \infty$. Hint: For a deceptively slick proof, explain why $\{|X_{n(k)} - X| > \delta\} \to 0$ almost surely then appeal to DC.

(iv) Conclude that $\mathbb{P}\{|X_n - X| > \delta\} \to 0$ for each $\delta > 0$.

*[2] Let $\mathcal{F}$ be the Borel sigma-field on the set $\Omega = [-1/2, 1/2]$ equipped with $\mathbb{P} =$ Lebesgue measure. Let $\mathcal{A}$ denote the sub-sigma-field of all symmetric Borel sets. That is, a Borel set is in $\mathcal{A}$ iff and only if $\{\omega \in \Omega : \omega \in A\} = \{\omega \in \Omega : -\omega \in A\}$.

(i) Show that a function f in $\mathcal{M}^+(\Omega, \mathcal{F})$ is $\mathcal{A}$-measurable if and only if $f(\omega) = f(-\omega)$ for all ω.

(ii) For each X in $\mathcal{M}^+(\Omega, \mathcal{F})$, show that $\mathbb{P}_{\mathcal{A}} X = (X(\omega) + X(-\omega))/2$ almost surely. Hint: $\int_{-1/2}^{1/2} f(x)\,dx = \int_{-1/2}^{1/2} f(-x)\,dx$.

*[3] Suppose $\mathbb{P}$ and $\mathbb{P}_\theta$, for $\theta \in \Theta$, are probability measures defined on a sigma-field $\mathcal{F}$, for some index set Θ. Suppose also that $\mathcal{G}$ is a sub-sigma-field of $\mathcal{F}$ and that there exist versions of densities

$$\frac{d\mathbb{P}_\theta}{d\mathbb{P}} = g_\theta(\omega)h(\omega) \qquad \text{with } g_\theta \in \mathcal{M}^+(\Omega, \mathcal{G}) \text{ for each } \theta$$

for a fixed $h \in \mathcal{M}^+(\Omega, \mathcal{F})$ that doesn't depend on θ. Define H to be a version of $\mathbb{P}_{\mathcal{G}} h$. [That is, choose one H from the $\mathbb{P}$-equivalence class of possibilities.]

(i) Show that $\mathbb{P}_\theta\{H = 0\} = 0$ for each θ.

(ii) Show that $\mathbb{P}_\theta\{H = \infty\} = 0$ for each θ. Hint: From $\infty > \mathbb{P}_\theta\{H = \infty\}$ deduce that $g_\theta\{H = \infty\} = 0$ a.s.[$\mathbb{P}$].

(iii) For each X in $\mathcal{M}^+(\Omega, \mathcal{F})$ and some fixed choice of γ of $\mathbb{P}_{\mathcal{G}}(Xh)$ define

$$Y(\omega) = \frac{\gamma(\omega)}{H(\omega)}\{0 < H < \infty\}.$$

Show that $\mathbb{P}_\theta(X \mid \mathcal{G}) = Y$ a.s.[$\mathbb{P}_\theta$] for every θ.