Statistics 330b/600b, Math 330b spring 2014
Solutions to sheet 10

[5] *Suppose $X_0 \in \mathcal{L}^2(\Omega, \mathcal{F}, \mathbb{P})$. Suppose also that $\mathcal{A}_1$ and $\mathcal{A}_2$ are sub-σ-fields of $\mathcal{F}$. Define $\mathcal{A} = \mathcal{A}_1 \cap \mathcal{A}_2$. (Do not assume that $\mathcal{A}$ is equal to an $\mathcal{A}_i$. That is, the $\mathcal{A}_i$'s are not nested.) Define sequences of $\mathcal{A}_1$-measurable random variables $\{Y_n : n \in \mathbb{N}_0\}$ and $\mathcal{A}_2$-measurable random variables $\{X_n : n \in \mathbb{N}\}$ recursively, by $Y_n = \mathbb{P}_{\mathcal{A}_1} X_n$ and $X_{n+1} = \mathbb{P}_{\mathcal{A}_2} Y_n$. Suppose there exists a Z in $\mathcal{L}^2(\Omega, \mathcal{F}, \mathbb{P})$ for which $\mathbb{P}|X_n - Z|^2 \to 0$. Show that $Z = \mathbb{P}_{\mathcal{A}} X_0$ almost surely. [I would also like to know when the X_n's converge in $\mathcal{L}^2$, but that seems a bit hard without using some sort of compactness property. Compare with Breiman and Friedman 1985.]*

As stated, the Problem was incorrect, but something similar is correct.

First note that, by HW9.1, there exists a real-valued $\mathcal{A}_2$-measurable random variable Z_2 to which some subsequence $X_{n'}$ converges almost surely. Argue again as in HW9.1 to get a sub-subsequence $X_{n''}$ that converges almost surely to Z. Deduce that $Z = Z_2$ almost surely.

The projection interpretation of conditional expectations in $\mathcal{L}^2$ shows that Y_n is orthogonal to $X_n - Y_n$ and that X_{n+1} is orthogonal to $Y_n - X_{n+1}$, which leads to

$$\mathbb{P}X_n^2 = \mathbb{P}(X_n - Y_n)^2 + \mathbb{P}Y_n^2 = \mathbb{P}(X_n - Y_n)^2 + \mathbb{P}(Y_n - X_{n+1})^2 + \mathbb{P}X_{n+1}^2.$$

From the convergence $\mathbb{P}X_n^2 \to \mathbb{P}Z^2$ it then follows that $\mathbb{P}(X_n - Y_n)^2 \to 0$ so that Y_n also converges in $\mathcal{L}^2$ to Z.

Repeat the argument from the second paragraph with X_n replaced by Y_n to deduce the existence of an $\mathcal{A}_1$-measurable random variable Z_1 for which $Y_{m'} \to Z_1$ and $Z_1 = Z$ almost surely.

At this point I made an error with negligible sets to conclude that Z must be almost surely equal to a random variable W that is measurable with respect to both $\mathcal{A}_1$ and $\mathcal{A}_2$. The W would then be a version of $\mathbb{P}_{\mathcal{A}} X_0$. (You need to check that $\mathbb{P}X_{n+1}A = \mathbb{P}Y_n A = \cdots = \mathbb{P}X_0 A$ for all $A \in \mathcal{A}$ then argue that $\mathbb{P}X_n A \to \mathbb{P}ZA$.) The conclusion is valid if both $\mathcal{A}_1$ and $\mathcal{A}_2$ contain $\mathcal{N} = \{F \in \mathcal{F} : \mathbb{P}F = 0\}$. Without that extra assumption the conclusion can be false, as shown by the following counterexample due to Oanh Nguyen and Daniel Montealegre.

Let $\mathbb{P}$ be Lebesgue measure on $\mathcal{B}[0,1]$, with $\Omega = [0,1]$. Define $A_1 = [0,1/2]$ and $A_2 = A_1 \cup \{1\}$. Define $\mathcal{A}_i = \{\emptyset, \Omega, A_i, A_i^c\}$ for $i = 1, 2$. Then $\mathcal{A} = \mathcal{A}_1 \cap \mathcal{A}_2 = \{\emptyset, \Omega\}$. Let X_0 be the indicator function of A_2.

By construction, the only $\mathcal{A}_1$ measurable random variable satisfying the defining properties of $\mathbb{P}_{\mathcal{A}_1} X_0$ is (the indicator function of) A_1. Similarly the only choice for $\mathbb{P}_{\mathcal{A}_2} A_1$ is A_2. It follows that $X_n = A_2$ and $Y_n = A_1$ for all n. The random variable Z must be equal to A_2 almost surely. Compare with the fact that the only choice for $\mathbb{P}_{\mathcal{A}} X_0$ is the constant function $1/2$.

References

Breiman, L. and J. H. Friedman (1985). Estimating optimal transformations for multiple regression and correlation. Journal of the American Statistical Association 80*(391), 580–598.*