Statistics 330b/600b, Math 330b spring 2014
Solutions to sheet 2

[4] *Define $\overline{\mathcal{M}}^+ = \overline{\mathcal{M}}^+(\mathcal{X},\mathcal{A},\mu)$ to consist of all those functions f mapping $\mathcal{X}$ into $[0,\infty]$ for which there exists $g,h \in \mathcal{M}^+ = \mathcal{M}^+(\mathcal{X},\mathcal{A},\mu)$ with $g(x) \le f(x) \le g(x)+h(x)$ for all x and $\mu h = 0$. You should NOT assume that $f \in \mathcal{M}^+$. Call the pair $g, g+h$ an $\mathcal{M}^+$-sandwich for f.*

(i) *Show that there is no ambiguity in defining $\overline{\mu} : \overline{\mathcal{M}}^+ \to [0,\infty]$ by $\overline{\mu} f = \mu g$ for an arbitrarily chosen $\mathcal{M}^+$-sandwich for f. That is, show that if g_1, g_1+h_1 and g_2, g_2+h_2 are both sandwiches for f then $\mu g_1 = \mu g_2$.*

For g_i and h_i as given, argue that $g_1 \le f \le g_2 + h_2$ so that $\mu g_1 \le \mu g_2 + \mu h_2 = \mu g_2$. Argue similarly for $\mu g_2 \le \mu g_1$.

(ii) *Define $\overline{\mathcal{A}} := \{D \subseteq \mathcal{X} : \mathbf{1}_D \in \overline{\mathcal{M}}^+\}$. Show that $\overline{\mathcal{A}}$ is a sigma-field with $\overline{\mathcal{A}} \supseteq \mathcal{A}$. Show also that if E is a subset of $\mathcal{X}$ for which $E \subseteq N$, for some $N \in \mathcal{N}_\mu$, then $E \in \overline{\mathcal{A}}$.*

For each $A \in \mathcal{A}$ the sandwich $A \le A \le A + 0$ gives $A \in \overline{\mathcal{A}}$. That is, $\mathcal{A} \subset \overline{\mathcal{A}}$. To prove that $\overline{\mathcal{A}}$ is a sigma-field it is enough to show it is stable under complements and countable unions.

For complements, if $g \le \overline{A} \le g + h$ is a sandwich for $\overline{A} \in \overline{\mathcal{A}}$ then by assumption $0 \le g \le 1$. We may also replace h by $h \wedge 1$ and still have $A \le g + h$. Why? Thus there are no $\infty - \infty$ problems in showing that

$$1 - g - h \le A^c \le 1 - g - h + h$$

Thus $G := (1-g-h)^+$ and $G + h$ provide a sandwich for A^c.

For countable unions, suppose $g_i \le \overline{A}_i \le g_i + h_i$ is a sandwich for each i in $\mathbb{N}$. Then $G := \sup_i g_i \le \cup_i \overline{A}_i \le G + H$, with $H = \sum_i h_i$ is a sandwich for the union.

For $E \subseteq N$ use the sandwich $0 \le E \le N$.

(iii) *Show that $\overline{\mathcal{M}}^+ = \mathcal{M}^+(\mathcal{X}, \overline{\mathcal{A}})$. (Sorry for the typo.)*

The set $\mathcal{E} := \{[t,\infty] : 0 \le t < \infty\}$ generates $\mathcal{B}[0,\infty]$. To show that each f in $\overline{\mathcal{M}}^+$ is $\overline{\mathcal{A}}$-measurable it suffices to show that $\{f \ge t\} \in \overline{\mathcal{A}}$ for each $0 \le t < \infty$. If $g \le f \le g + h$ is a sandwich for f then

$$\{g \ge t\} \le \{f \ge t\} \le \{g \ge t\} + \infty\{h > 0\} \text{ is a sandwich for } \{f \ge t\}.$$

Conversely, to show that $\mathcal{M}^+(\mathcal{X}, \overline{\mathcal{A}}) \subseteq \overline{\mathcal{M}}^+$, start with simple functions. Suppose $f = \sum_i \alpha_i \overline{A}_i \in \mathcal{M}^+_{\text{simple}}(\mathcal{X}, \overline{\mathcal{A}})$. By definition of $\overline{\mathcal{A}}$, each $\overline{A}_i$ has a sandwich $g_i \le \overline{A}_i \le g_i + h_i$, so that

$$G := \sum\nolimits_i \alpha_i g_{\le} \sum\nolimits_i \alpha_i \overline{A}_i \le G + \sum\nolimits_i \alpha_i h_i$$

is a sandwich for f.

Write the general f in $\mathcal{M}^+(\mathcal{X},\overline{\mathcal{A}})$ as a pointwise increasing limit of functions f_n in $\mathcal{M}^+_{\text{simple}}(\mathcal{X},\overline{\mathcal{A}})$ with sandwiches $g_n \le f_n \le g_n + h_n$. Then

$$G := \sup_i g_i \le f \le G + \sum\nolimits_i h_i$$

is a sandwich for f.

(iv) Show that $\overline{\mu}$ defines an increasing, linear functional on $\overline{\mathcal{M}}^+$ with the Monotone Convergence property.

If $f_1 \le f_2$ are functions in $\mathcal{M}^+$ with sandwiches $g_i \le f_i \le g_i + h_i$ then

$$g_1 \le G := g_1 \vee g_2 \le f_2 \le G + h_2$$

so that $\overline{\mu} f_1 = \mu g_1 \le \mu G = \overline{\mu} f_2$. If $\alpha_i \in \mathbb{R}^+$ and $f_i \in \overline{\mathcal{M}}^+$ with sandwiches $g_i \le f_i \le g_i + h_i$, for $i = 1, 2$, then

$$G := \alpha_1 g_1 + \alpha_2 g_2 \le \alpha_1 f_1 + \alpha_2 f_2 \le G + \alpha_1 h_1 + \alpha_2 h_2$$

is also a sandwich and

$$\overline{\mu}(\alpha_1 f_1 + \alpha_2 f_2) = \mu G = \alpha_1 \mu g_1 + \alpha_2 \mu g_2 = \alpha_1 \overline{\mu} f_1 + \alpha_2 \overline{\mu} f_2.$$

Finally, if a sequence f_n in $\overline{\mathcal{M}}^+$, with sandwiches $g_n \le f_n \le g_n + h_n$, increases pointwise to f then $G_n := \max_{i \le n} g_i$ increases pointwise to $G := \sup_{i \in \mathbb{N}} g_i$ then $G_n \le f_n \le G_n + H$, where $H = \sum_{i \in \mathbb{N}} h_i$, so that

$$\overline{\mu} f_n = \mu G_n \uparrow \mu G = \overline{\mu} f,$$

the last equality coming from the fact that $G \le f \le G + H$ is a sandwich for f.

(v) Show that the restriction of $\overline{\mu}$ to (the indicator functions of sets in) $\overline{\mathcal{A}}$ is a measure and that $\overline{\mu}$ is the integral with respect to that measure.

As several students pointed out, this part is just Theorem 2.13 of UGMTP.