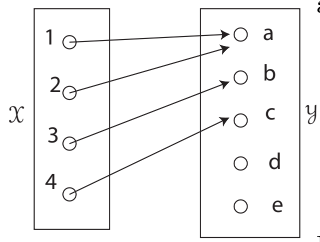
Statistics 330b/600b, Math 330b spring 2015
Homework # 1
Due: Thursday 22 January

Please attempt at least the starred problems. Please explain your reasoning.

*[1] Suppose T maps a set $\mathcal{X}$ into a set $\mathcal{Y}$. For each $B \subseteq \mathcal{Y}$ and $A \subseteq \mathcal{X}$ define $T^{-1}B := \{x \in \mathcal{X} : T(x) \in B\}$. and $T(A) := \{T(x) : x \in A\}$. Four of the following eight assertions are always true and four are not always true.



$$(i)\quad T\left(\cup_i A_i\right) = \cup_i T(A_i) \qquad (ii)\quad T^{-1}\left(\cup_i B_i\right) = \cup_i T^{-1}(B_i)$$
$$(iii)\quad T\left(\cap_i A_i\right) = \cap_i T(A_i) \qquad (iv)\quad T^{-1}\left(\cap_i B_i\right) = \cap_i T^{-1}(B_i)$$
$$(v)\quad T\left(A^c\right) = \left(T\left(A\right)\right)^c \qquad (vi)\quad T^{-1}\left(B^c\right) = \left(T^{-1}\left(B\right)\right)^c$$
$$(vii)\quad T^{-1}\left(T(A)\right) = A \qquad (viii)\quad T\left(T^{-1}(B)\right) = B$$

Provide counterexamples for each of the false assertions. (Hint: All the counterexamples can be constructed using the special case shown in the picture.)

*[2] Suppose x and y are two distinct points in a set $\mathcal{X}$ and $\mathcal{E}$ is a set of subsets of $\mathcal{X}$ with the property that $\mathbf{1}_E\{x\} = \mathbf{1}_E\{y\}$ for all $E \in \mathcal{E}$. Show that $\mathbf{1}_B\{x\} = \mathbf{1}_B\{y\}$ for all $B \in \sigma(\mathcal{E})$. Hint: Show that $\mathcal{B}_0 = \{B \in \sigma(\mathcal{E}) : \mathbf{1}_B\{x\} = \mathbf{1}_B\{y\}\}$ is a sigma-field.

*[3] Let $\mathcal{G}$ denote the set of all open subsets of $\mathbb{R}^2$ and $\mathcal{R}$ denote the set of all closed rectangles in $\mathbb{R}^2$ of the form $[a_1, a_2] \times [b_1, b_2]$ with a_1, a_2, b_1 and b_2 all rational. Show that $\sigma(\mathcal{G}) = \sigma(\mathcal{R})$.

[4] How many different sigma-fields are there on the set $\mathcal{X} = \{1, 2, 3, 4, 5\}$?

[5] Let $A_1, \dots, A_N$ be events in a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. For each subset J of $\{1, 2, \dots, N\}$ write A_J for $\cap_{i\in J} A_i$. Define $S_k := \sum_{|J|=k} \mathbb{P}A_J$, where $|J|$ denotes the number of indices in J. For $0 \le m \le n$ define

$$B_m = \{\text{exactly } m \text{ of the } A_i\text{'s occur}\} = \{\omega \in \Omega : \textstyle\sum_{i=1}^N \mathbf{1}_{A_i}(\omega) = m\}.$$

(i) Explain why

$$\mathbf{1}_{B_m} = \sum\nolimits_{|J|=m} \prod\nolimits_{i\in J} \mathbf{1}_{A_i} \prod\nolimits_{j\in J^c} (1 - \mathbf{1}_{A_j}).$$

(ii) Deduce that

$$\mathbf{1}_{B_m} = \sum\nolimits_{\ell=m}^N (-1)^{\ell-m} \sum\nolimits_{|K|=\ell} \binom{\ell}{m} \mathbf{1}_{A_K}.$$

(iii) Take expectations to deduce that

$$\mathbb{P}B_m = \binom{m}{m} S_m - \binom{m+1}{m} S_{m+1} + \dots \pm \binom{N}{m} S_N.$$

Compare with the method suggested in UGMTP Problem 1.1. (You may use that method if you prefer it.)