Statistics 330b/600b, Math 330b spring 2015
Homework # 10
Due: Thursday 9 April

*[1] Let $\{(Z_n, \mathcal{F}_n) : n \in \mathbb{N}_0\}$ be a submartingale and τ be a stopping time. Define $X_n = Z_{\tau\wedge n}$. Show that $\{(X_n, \mathcal{F}_n) : n \in \mathbb{N}_0\}$ is also a submartingale. Hint: For F in $\mathcal{F}_{n-1}$, consider separately the contributions to $\mathbb{P}X_nF$ and $\mathbb{P}X_{n-1}F$ from the regions $\{\tau \le n-1\}$ and $\{\tau \ge n\}$.

*[2] Let τ be a stopping time for a filtration $\{\mathcal{F}_i : 0 \le i \le N\}$, with $0 \le \tau \le N$. For an integrable random variable X, define $X_i := \mathbb{P}(X \mid \mathcal{F}_i)$. Show that

$$\mathbb{P}(X \mid \mathcal{F}_\tau) = \sum\nolimits_{i=0}^{N} \{\tau = i\} X_i = X_\tau \qquad \text{almost surely.}$$

*[3] (Birnbaum and Marshall, 1961) Let $0 = X_0, X_1, \ldots, X_N$ be nonnegative integrable random variables that are adapted to a filtration $\{\mathcal{F}_i : 0 \le i \le N\}$. Suppose there exist constants θ_i, with $0 \le \theta_i \le 1$, for which

$$(*) \qquad \mathbb{P}(X_i \mid \mathcal{F}_{i-1}) \ge \theta_i X_{i-1} \qquad \text{for } 0 \le i \le N.$$

(Interpret this inequality to mean that there exist nonnegative, $\mathcal{F}_{i-1}$-measurable random variables Y_{i-1} for which $\mathbb{P}(X_i \mid \mathcal{F}_{i-1}) = Y_{i-1} + \theta_i X_{i-1}$ almost surely.) Let $C_1 \ge C_2 \ge \cdots \ge C_{N+1} = 0$ be constants. Prove the inequality

$$(**) \qquad \mathbb{P}\{\max\nolimits_{i \le N} C_i X_i \ge 1\} \le \sum\nolimits_{i=1}^{N} (C_i - \theta_{i+1} C_{i+1}) \mathbb{P} X_i,$$

by following these steps.

(i) Define $\eta_i = W_i - W_{i-1}$, where $W_i = C_i X_i$. (Thus $W_i = \eta_1 + \cdots + \eta_i$.) Show that $\mathbb{P}(\eta_i \mid \mathcal{F}_{i-1}) \le C_i Y_{i-1}$ almost surely, for $1 \le i \le N$.

(ii) Deduce that $W_i \le M_i + A_i$, where $\{(M_i, \mathcal{F}_i)\}$ is a martingale with $M_0 = 0$ and $A_i := \sum_{j=1}^{i} C_j Y_{j-1}$ is an increasing sequence.

(iii) Use the Stopping Time Lemma to show that the left-hand side of inequality $(**)$ is less than $\mathbb{P}A_N$, then rearrange the sum for $\mathbb{P}A_N$ to get the asserted upper bound.

[4] (Hard) Suppose τ is a stopping time for the natural filtration $\{\mathcal{F}_i : i \in \mathbb{N}_0\}$ generated by real-valued random variables $\{X_i : i \in \mathbb{N}_0\}$ on a set Ω. (That is, $\mathcal{F}_i$ is the smallest sigma-field on Ω for which X_j is $\mathcal{F}_i\backslash\mathcal{B}(\mathbb{R})$-measurable for $0 \le j \le i$.) Define new random variables $Z_i = X_{\tau\wedge i}$, for $i \in \mathbb{N}_0$. Define $\mathcal{G}$ to be the smallest sigma-field on Ω for which each Z_i, for $i \in \mathbb{N}_0$, is $\mathcal{G}\backslash\mathcal{B}(\mathbb{R})$-measurable. Show that $\mathcal{G} = \mathcal{F}_\tau$.

References

Birnbaum, Z. W. and A. W. Marshall (1961). Some multivariate Chebyshev inequalities with extensions to continuous parameter processes. *Annals of Mathematical Statistics 32*(3), 687–703.