**Statistics 330b/600b, Math 330b spring 2015**
Homework # 11
Due: Thursday 16 April

*[1] Suppose $\{(X_i, \mathcal{F}_i) : i \in \mathbb{N}_0\}$ is a martingale (on $\Omega, \mathcal{F}, \mathbb{P}$) with $\sup_i \mathbb{P}X_i^2 = K < \infty$. By the following steps show that $X_i$ converges almost surely and in $\mathcal{L}^2$ to some random variable $Z$ in $\mathcal{L}^2(\Omega, \mathcal{F}_\infty, \mathbb{P})$, where $\mathcal{F}_\infty = \sigma\left(\cup_{i\in\mathbb{N}_0}\mathcal{F}_i\right)$.

(i) For $i \geq 1$ define $\xi_i = X_i - X_{i-1}$. Show that $\sum_{i\in\mathbb{N}_0} \mathbb{P}\xi_i^2 < \infty$. Deduce that $\{X_i\}$ is Cauchy sequence, which converges in $\mathcal{L}^2$ to some $Z$ in $\mathcal{L}^2(\Omega, \mathcal{F}_\infty, \mathbb{P})$.

(ii) For $m < n$ define $\Delta_{m,n} = \max_{m<i<n} |X_i - X_m|$ and $\Delta_{m,\infty} = \sup_{i>m} |X_i - X_m|$. Prove that

$$\mathbb{P}\Delta_{m,n}^2 \leq \delta_m^2 := \sum\nolimits_{i>m} \mathbb{P}\xi_i^2 \to 0 \qquad \text{as } m \to \infty.$$

Deduce that $\|\Delta_{m,\infty}\|_2 \leq \delta_m^2$.

(iii) Define $D_m = \sup_{i>m} |X_i - Z|$. Show that

$$\|D_m\|_2 \leq \delta_m + \|X_m - Z\|_2 \to 0 \qquad \text{as } m \to \infty.$$

(iv) Deduce that $\mathbb{P}\limsup_i |X_i - Z|^2 = 0$ and $|X_i - Z| \to 0$ almost surely.

[2] Use Konecker's lemma (UGMTP Problem 4.22) and Problem [1] to reprove SLLN2: for independent random variables $\{\xi_i : i \in \mathbb{N}\}$ with $\sum_i \mathbb{P}\xi_i^2 < \infty$ and $\mathbb{P}\xi_i = 0$ for each $i$,

$$n^{-1}\sum\nolimits_{i\leq n} \xi_i \to 0 \qquad \text{almost surely.}$$

Hint: Consider $X_n = \sum_{I\leq n} \xi_i/i$.

*[3] Let $\mathcal{F}$ be a countably generated sigma-field on a set $\Omega$, that is, $\mathcal{F} = \sigma(\mathcal{E})$, where $\mathcal{E} = \{E_i : i \in \mathbb{N}\}$. Define $\mathcal{F}_k = \sigma\{E_1, \dots, E_k\}$. Note that $\mathcal{F} = \sigma\left(\cup_{k\in\mathbb{N}}\mathcal{F}_k\right)$.

(i) Show that there is a finite partition $\pi_k$ of $\Omega$ into disjoint $\mathcal{F}_k-$measurable sets for which $\mathcal{F}_k = \sigma(\pi_k)$. Also show that a real-valued function $g$ is $\mathcal{F}_k$-measurable iff $g$ is constant on each $A$ in $\pi_k$.

(ii) If $f \in \mathcal{L}^1(\Omega, \mathcal{F}, \mathbb{P})$ show that

$$\mathbb{P}_{\mathcal{F}_k} f = f_k(\omega) := \sum\nolimits_{A\in\pi_k} \{\omega \in A\}\mathbb{P}_A f \qquad \text{where } \mathbb{P}_A f = \{\mathbb{P}A > 0\}\mathbb{P}(fA)/\mathbb{P}A.$$

(iii) Show that $\{(f_k, \mathcal{F}_k) : k \in \mathbb{N}\}$ is a martingale.

(iv) Now suppose $f \geq 0$ and $\mathbb{P}f < \infty$. Explain why $\{f_k\}$ converges almost surely to some nonnegative $f_\infty$ in $\mathcal{L}_1(\Omega, \mathcal{F}, \mathbb{P})$.

(v) For each $m \in \mathbb{N}$ define $f_{k,m} = \mathbb{P}_{\mathcal{F}_k}(m \wedge f)$. Show that $\{f_{k,m} : k \in \mathbb{N}\}$ converges almost surely and in $\mathcal{L}^1$ to some nonnegative $f_{\infty,m}$ in $\mathcal{L}^1(\omega, \mathcal{F}, \mathbb{P})$.

(vi) Show that $\mathbb{P}f = \mathbb{P}f_k \geq \mathbb{P}f_{k,m} = \mathbb{P}(m \wedge f)$ for each $m$. Deduce that $\mathbb{P}f_k \to \mathbb{P}f$ and $\mathbb{P}|f_k - f| \to 0$.

(vii) Show that $\mathbb{P}f_\infty E = \mathbb{P}fE$ for each $E \in \cup_k\mathcal{F}_k$. Hint: Consider $E \in \mathcal{F}_\ell$ and $\mathbb{P}f_kE$ for $k \geq \ell$. Deduce that $f_\infty = f$ almost surely.