Statistics 330b/600b, Math 330b spring 2015
Homework # 12
Due: Thursday 23 April

*[1] Let $(\mathcal{Y}, d)$ be a metric space with a countable dense subset $\{y_i : i \in \mathbb{N}\}$. For a fixed $\delta > 0$ define functions $g_0 = \delta$ and $g_i(y) = (1 - d(y, y_i)/\delta)^+$. Define

$$G_N(y) := \sum\nolimits_{0 \le i \le N} g_i(y) \quad \text{AND} \quad \ell_{i,N}(y) = g_i(y)/G_N(y) \qquad \text{for } 0 \le i \le N.$$

Note that $\sum_{0 \le i \le N} \ell_{i,N}(y) = 1$ for all y.

(i) Show that $\ell_{i,N} \in \mathrm{BL}(\mathcal{X})$ for all i and N.

(ii) Show that $\ell_{0,N}(y) \downarrow L_\delta(y)$ as $N \to \infty$, with $L_\delta(y) \le \delta$ for each y.

*[2] Suppose $(\mathcal{X}, d_\mathcal{X})$ and $(\mathcal{Y}, d_\mathcal{Y})$ are both separable metric spaces. Define a metric on $\mathcal{X} \times \mathcal{Y}$ by $d\left((x_1, y_1), (x_2, y_2)\right) := \max\left(d_\mathcal{X}(x_1, x_2), d_\mathcal{Y}(y_1, y_2)\right)$. Suppose also that we have measurable functions $X_n : (\Omega_n, \mathcal{F}_n) \to (\mathcal{X}, \mathcal{B}(\mathcal{X}))$ and $Y_n : (\Omega_n, \mathcal{F}_n) \to (\mathcal{Y}, \mathcal{B}(\mathcal{Y}))$ for which

$$\mathbb{P}_n g(X_n) h(Y_n) \to \mu^{x,y} g(x) h(y) \qquad \text{for all } g \in \mathrm{BL}(\mathcal{X}) \text{ and } h \in \mathrm{BL}(\mathcal{Y}),$$

where μ is some probability measure on $\mathcal{B}(\mathcal{X} \times \mathcal{Y})$. Follow these steps to show that $(X_n, Y_n) \rightsquigarrow \mu$.

(i) Fix a $\delta > 0$ and an f in $\mathrm{BL}(\mathcal{X} \times \mathcal{Y}, d)$. With $\ell_{i,N}$ as in Problem [1], define $f_N(x, y) = \sum_{1 \le i \le N} f(x, y_i) \ell_{i,N}(y)$. For each f in $\mathrm{BL}(\mathcal{X} \times \mathcal{Y}, d)$ show that

$$|f(x, y) - f_N(x, y)| \le \|f\|_{\mathrm{BL}} \left(\delta + \ell_{0,N}(y)\right).$$

(ii) Deduce that $\limsup_n |\mathbb{P}_n f(X_n, Y_n) - \mu f| \le 2 \|f\|_{\mathrm{BL}} \left(\delta + \mu \ell_{0,N}(y)\right)$.

(iii) Complete the proof: show $(X_n, Y_n) \rightsquigarrow \mu$.

(iv) For independent X_n and Y_n with $X_n \rightsquigarrow P$ and $Y_n \rightsquigarrow Q$, prove $(X_n, Y_n) \rightsquigarrow P \otimes Q$.

[3] Let $\{X_{n,i}\}$ be a triangular array of random variables, independent within each row and satisfying

(a) $\max_i |X_{n,i}| \to 0$ in probability,

(b) $\sum_i \mathbb{P} X_{n,i}\{|X_{n,i}| \le \epsilon\} \to \mu$ for each $\epsilon > 0$,

(c) $\sum_i \mathrm{var}\big(X_{n,i}\{|X_{n,i}| \le \epsilon\}\big) \to \sigma^2 < \infty$ for each $\epsilon > 0$.

(i) Show that there exists a sequence of positive numbers $\{\epsilon_n\}$ that tends to zero slowly enough that

(d) $\mathbb{P}\{\max_i |X_{n,i}| > \epsilon_n\} \to 0$,

(e) $\sum_i \mathbb{P} X_{n,i}\{|X_{n,i}| \le \epsilon_n\} \to \mu$,

(f) $\sum_i \mathrm{var}\big(X_{n,i}\{|X_{n,i}| \le \epsilon_n\}\big) \to \sigma^2$.

(ii) Deduce that $\sum_i X_{n,i} \rightsquigarrow N(\mu, \sigma^2)$. Hint: Consider $\eta_{n,i} := X_{n,i}\{|X_{n,i}| \le \epsilon_n\}$ and $\xi_{n,i} := \eta_{n,i} - \mathbb{P}\eta_{n,i}$.

*[4] Suppose $\{X_n\}$ is a sequence of real-valued random variables and P is a probability measure on $\mathcal{B}(\mathbb{R})$ for which $\mathbb{P}_n\{X_n \le t\} \to P(-\infty, t]$ for each t with $P\{t\} = 0$. Show that $X_n \rightsquigarrow P$. Hint: Approximate an f in $\text{BL}(\mathbb{R})$ by a function of the form $f_N(x) = \sum_{j=1}^{N} f(x_j)\{x_j < x \le x_{j+1}\}$, for a suitably chosen set of P-continuity points $-\infty < x_1 < x_2 < \cdots < x_N < x_{N+1} < \infty$.

[5] [Warning: very messy.] In class I proved: if $X_n \rightsquigarrow P$ and $d(X_n, Y_n) \le \Delta_n \to 0$ in probability then $Y_n \rightsquigarrow P$. I showed that $\mathbb{P}_n f(Y_n) \to Pf$ for each $f \in \text{BL}(\mathcal{X})$, starting from $\mathbb{P}_n f(X_n) \to Pf$. Repeat the proof assuming only that f is just bounded and continuous and $\mathbb{P}_n f(X_n) \to Pf$.

Please do not use any of the facts I established using the BL *definition. The point is to show you the advantages of starting from* BL *rather than from the more traditional approach using bounded, continuous functions.*

[6] Suppose H is a continuously differentiable function on $\mathbb{R}$ which is zero outside some bounded interval. For a given bounded measurable function f on $\mathbb{R}$, define $g(x) := \int f(y)H(x-y)\,dy$.

(i) Show (rigorously) that g is differentiable with $g'(x) = \int f(y)H'(x-y)\,dy$.

(ii) Explain why g belongs to $\mathcal{C}^\infty(\mathbb{R})$ if $H \in \mathcal{C}^\infty(\mathbb{R})$.