Statistics 330b/600b, Math 330b spring 2015
Homework # 2
Due: Thursday 29 January

Please attempt at least the starred problems. Please explain your reasoning.

*[1] Suppose μ is a countably additive measure defined on a sigma-field $\mathcal{A}$ on $\mathcal{X}$. Define $\mathcal{A}_\mu$ to be the set of all $D \subseteq \mathcal{X}$ for which there exists sets $A, B \in \mathcal{A}$ with $B \subseteq D \subset A$ and $\mu(AB^c) = 0$. Call A, B a μ-sandwich for D.

(i) If a set $D \in \mathcal{A}_\mu$ has two μ-sandwiches, A_1, B_1 and A_2, B_2, show that $\mu A_1 = \mu A_2 = \mu B_1 = \mu B_2$.

(ii) Show that $\mathcal{A}_\mu$ is a sigma-field with $\mathcal{A} \subseteq \mathcal{A}_\mu$.

(iii) Define $\widetilde{\mu}$ on $\mathcal{A}_\mu$ by $\widetilde{\mu} D = \mu B$ for any choice of μ-sandwich A, B for D. Show that $\widetilde{\mu}$ is a countably additive measure on $\mathcal{A}_\mu$ for which $\widetilde{\mu} D = \mu D$ if $D \in \mathcal{A}$.

*[2] Suppose $\mathcal{A}$ is a sigma-field on a set $\mathcal{X}$ and $\mathcal{B}$ is a sigma-field on a set $\mathcal{Y}$. Suppose also that T is an $\mathcal{A}\backslash\mathcal{B}$-measurable function from $\mathcal{X}$ to $\mathcal{Y}$. Let μ be a countably additive measure on $\mathcal{A}$.

(i) For each $h \in \mathcal{M}^+(\mathcal{X}, \mathcal{A})$ and $g \in \mathcal{M}^+(\mathcal{Y}, \mathcal{B})$ define $\nu_h(g) = \mu\left(h(x)g(Tx)\right)$. Show that each ν_h corresponds to an integral with respect to a countably additive measure on $\mathcal{B}$.

(ii) Suppose g is a function in $\mathcal{M}^+(\mathcal{Y}, \mathcal{B})$ for which $\nu_1(g) = 0$. (Here ν_1 is the ν_h for h equal to the constant function 1.) Show that $\nu_h g = 0$ for all $h \in \mathcal{M}^+(\mathcal{X}, \mathcal{A})$.

*[3] Suppose $f_1, \dots, f_k \in \mathcal{M}^+(\mathcal{X}, \mathcal{A})$ and $\theta_1, \dots, \theta_k$ are strictly positive numbers that sum to one. Let μ be a measure on $\mathcal{A}$. Show that $\mu \prod_{i \le k} f_i^{\theta_i} \le \prod_{i \le k} (\mu f_i)^{\theta_i}$ by following these steps.

(i) Explain why the inequality is trivially true if $\mu f_i = 0$ for at least one i or if $\mu f_i = \infty$ for at least one i (and all the other μf_j are strictly positive).

(ii) Explain why there is no loss of generality in assuming that $\mu f_i = 1$ for each i and $f_i(x) < \infty$ for each x and i.

(iii) For all $a_1, \dots, a_k \in \mathbb{R}^+$, show that $\prod_{i \le k} a_i^{\theta_i} \le \sum_{i \le k} \theta_i a_i$. Hint: First dispose of the trivial case where at least one a_i is zero, then rewrite the inequality using $b_i = \log a_i$. *You do not need to reprove that the* log *function is concave on* $(0, \infty)$.

(iv) Complete the proof by considering the inequality from (iii) with $a_i = f_i(x)$.

[4]

(i) For all nonnegative real numbers $a_1, \dots, a_n$ show that

$$\sum_i a_i \le \max_i a_i + \sum_{i<j} \min(a_i, a_j) \le \sum_i a_i + \sum_{i<j<k} \min(a_i, a_j, a_k).$$

Hint: First explain why, without loss of generality, you may assume that $a_1 \ge a_2 \ge \cdots \ge a_n \ge 0$.

(ii) Suppose $f_1, \dots, f_n$ are functions in $\mathcal{M}^+(\mathcal{X}, \mathcal{A})$ and μ is a measure on $\mathcal{A}$ for which $\mu f_i < \infty$ for each i. Show that

$$\begin{aligned} \sum_i \mu f_i - \sum_{i<j} \mu(f_i \wedge f_j) &\le \mu\left(\max_i f_i\right) \\ &\le \sum_i \mu f_i - \sum_{i<j} \mu(f_i \wedge f_j) + \sum_{i<j<k} \mu(f_i \wedge f_j \wedge f_k). \end{aligned}$$

Here $\wedge$ denotes pointwise minimum. Note: Expressions like $f_1(x) - f_2(x) \wedge f_3(x)$ might not be well defined for all x.