Statistics 330b/600b, Math 330b spring 2015
Homework # 3
Due: Thursday 5 February

Please attempt at least the starred problems. Please explain your reasoning. If you solve Problem [2] you can skip Problem [1] (but you might find some clues by first looking at the easier Problem).

*[1] *For f in $\mathcal{L}^1 := \mathcal{L}^1(\mathcal{X}, \mathcal{A}, \mu)$ define $\|f\|_1 = \mu|f|$. Let $\{f_n\}$ be a Cauchy sequence in $\mathcal{L}^1$, that is, $\|f_n - f_m\|_1 \to 0$ as $\min(m,n) \to \infty$. Show that there exists an f in $\mathcal{L}^1$ for which $\|f_n - f\|_1 \to 0$, by following these steps. Note: Don't confuse Cauchy sequences (in $\mathcal{L}^1$ distance) of functions with Cauchy sequences of real numbers.*

(i) *For each $k \in \mathbb{N}$ there exists an $n(k) \in \mathbb{N}$ for which: $\|f_n - f_m\|_1 < 2^{-k}$ when $\min(m,n) \geq n(k)$. Without loss of generality assume that $n(k)$ is strictly increasing with k. Define $H(x) := \sum_{k=1}^{\infty} |f_{n(k)}(x) - f_{n(k+1)}(x)|$. Show that $\mu H < \infty$.*

(ii) *Define $g_k = f_{n(k)}$. Show that $\{g_k(x) : k \in \mathbb{N}\}$ is a Cauchy sequence of real numbers for each x in the set $\{H < \infty\}$. Explain why*

$$f(x) = \liminf_{k\to\infty} g_k(x)\{H(x) < \infty\}$$

is a real-valued, $\mathcal{A}$-measurable function for which

$$H(x) \geq |g_k(x) - f(x)| \to 0 \qquad \text{a.e.}[\mu].$$

(iii) *Deduce that $\|g_k - f\|_1 \to 0$ as $k \to \infty$.*

(iv) *Show that f belongs to $\mathcal{L}^1(\mu)$ and $\|f_n - f\|_1 \to 0$ as $n \to \infty$.*

[2] *Let $(\mathcal{X}, \mathcal{A}, \mu)$ be a measure space with $\mu\mathcal{X} < \infty$. Write $\mathcal{R}$ for the set of all $\mathcal{A}\backslash\mathcal{B}(\mathbb{R})$-measurable functions from $\mathcal{X}$ into $\mathbb{R}$.*

(i) *Define $\psi(t) = 1 - e^{-t}$ for $0 \leq t < \infty$ and $\psi(\infty) = 1$. For $f, g \in \mathcal{R}$ define $d(f,g) = \mu\psi(|f - g|)$. Show that d is a semi-metric on $\mathcal{R}$ for which: $d(f,g) = 0$ iff $f = g$ a.e.$[\mu]$; and $d(f_n, 0) \to 0$ iff $\mu\{x : |f_n(x)| > \epsilon\} \to 0$ for every $\epsilon > 0$.*

(ii) *Show that d is complete: if $\{f_n : n \in \mathbb{N}\} \subset \mathcal{R}$ and $d(f_n, f_m) \to 0$ as $\min(m,n) \to \infty$ then there exists an $f \in \mathcal{R}$ for which $d(f_n, f) \to 0$.*

*[3] *For each θ in $[0,1]$ define $\mu_{\theta,n}$ to be the Binomial(n,θ) distribution. That is,*

$$\mu_{\theta,n} f = \sum\nolimits_{k=0}^{n} f(k) \binom{n}{k} \theta^k (1-\theta)^{n-k}.$$

You may assume without proof that $\int x\,d\mu_{\theta,n} = n\theta$ and $\int (x - n\theta)^2 d\mu_{\theta,n} = n\theta(1-\theta)$.

Let g be a continuous function defined on $[0,1]$. Remember that g must also be uniformly continuous: for each fixed $\epsilon > 0$ there exists a $\delta_\epsilon > 0$ such that

$$|g(s) - g(t)| \leq \epsilon \qquad \text{whenever } |s - t| \leq \delta_\epsilon, \text{ for } s,\ t \text{ in } [0,1].$$

Remember also that $|g|$ must be uniformly bounded, say, $\sup_t |g(t)| = M < \infty$.

(i) *Show that $p_n(\theta) := \mu_{\theta,n} g(x/n)$ is a polynomial in θ.*

(ii) *Show that $|g(x/n) - g(\theta)| \leq \epsilon + 2M|x - n\theta|^2/(n\delta_\epsilon)^2$ for $0 \leq x \leq n$.*

(iii) *Deduce that $\sup_{0\leq\theta\leq 1} |p_n(\theta) - g(\theta)| < 2\epsilon$ for n large enough. That is, deduce that $g(\cdot)$ can be uniformly approximated by polynomials over the range $[0,1]$, a result known as the **Weierstrass approximation theorem**.*