**Statistics 330b/600b, Math 330b spring 2015**
Homework # 4
Due: Thursday 12 February

*For Problems [2] and [4], $(\mathcal{X}, \mathcal{A}, \mu)$ is a measure space and $\Psi$ is a convex, increasing function for which $\Psi(0) = 0$ and $\Psi(x) \to \infty$ as $x \to \infty$. For convenience, define $\Psi(\infty) = \infty$. The set $\mathcal{L}^\Psi = \mathcal{L}^\Psi(\mathcal{X}, \mathcal{A}, \mu)$ is defined to be the set of all real-valued $\mathcal{A}$-measurable functions $f$ on $\mathcal{X}$ for which there exists some positive real $c_0$ (depending on $f$) such that $\mu\Psi(|f|/c_0) < \infty$. For $f$ in $\mathcal{L}^\Psi$*

$$\|f\|_\Psi := \inf\{c > 0 : \mu\Psi(|f|/c) \le 1\}.$$

*[1] Let $\{f_n : n \in \mathbb{N}\}$ be a sequence in $\mathcal{M}^+(\mathcal{X}, \mathcal{A})$ and $\mu$ be a measure on $\mathcal{A}$.

(i) Suppose $\lim_{n\to\infty} \mu\{x : f_n(x) > \epsilon\} = 0$ for each $\epsilon > 0$. Show that there exists an increasing sequence $\{n(k) : k \in \mathbb{N}\}$ for which $\sum_{k\in\mathbb{N}} \mu\{f_{n(k)} > 2^{-k}\} < \infty$. Deduce that $f_{n(k)} \to 0$ a.e.$[\mu]$.

(ii) Now suppose $\mu\mathcal{X} < \infty$ and $f_n \to 0$ a.e.$[\mu]$. Show that $\mu\{\sup_{i\ge n} f_i > \epsilon\} \to 0$ for each $\epsilon > 0$. Hint: Monotonicity.

*[2] Suppose $f \in \mathcal{L}^\Psi(\mathcal{X}, \mathcal{A}, \mu)$ and $c = \|f\|_\Psi > 0$. Show that $\mu\Psi(|f|/c) \le 1$. Hint: DC. For extra brownie points, find an example where the last inequality is strict, or prove that there must always be equality.

*[3] (Projections in Hilbert space, almost) Let $\mathcal{K}$ be a closed, convex subset of $\mathcal{H} := \mathcal{L}^2(\mathcal{X}, \mathcal{A}, \mu)$. For a fixed $f$ in $\mathcal{H}\backslash\mathcal{K}$ define $\delta := \inf\{\|f - h\|_2 : h \in \mathcal{K}\}$. In this problem you will show that there is an $f_0$ (unique up to $\mu$-equivalence) in $\mathcal{K}$ for which $\|f - f_0\|_2 = \delta$ and establish some related properties.

(i) For all $a, b \in \mathcal{H}$ show that

$$\|a + b\|_2^2 + \|a - b\|_2^2 = 2\|a\|_2^2 + 2\|b\|_2^2.$$

(ii) If $h_n \in \mathcal{K}$ are chosen to make $\delta_n := \|f - h_n\|_2 \to \delta$, show that $\{h_n : n \in \mathbb{N}\}$ is a Cauchy sequence in $\mathcal{H}$. Invoking completeness of $\mathcal{H}$, deduce that there exists some $f_0$ in $\mathcal{K}$ for which $\|h_n - f_0\|_2 \to 0$. Explain why $\|f - f_0\|_2 = \delta$.

(iii) For all $h \in \mathcal{K}$ and all $t \in [0, 1]$ explain why $\|f - (1 - t)f_0 - th\|_2 \ge \delta$. Deduce that $\mathcal{K}$ is contained in the closed halfspace $\{h \in \mathcal{H} : \langle h - f_0, f_0 - f\rangle \ge 0\}$.

(iv) If $\mathcal{K}$ is actually a closed subspace of $\mathcal{H}$, deduce that $\langle h, f_0 - f\rangle = 0$ for all $h$ in $\mathcal{K}$.

[4] Let $\{f_n\}$ be a Cauchy sequence in $\mathcal{L}^\Psi$, that is, $\|f_n - f_m\|_\Psi \to 0$ as $\min(m, n) \to \infty$. In particular, there exists an increasing sequence $\{n(k) : k \in \mathbb{N}\}$ such that $\|f_n - f_m\|_\Psi < \delta_k = 2^{-k}$ for $\min(m, n) \ge n(k)$.

Show that there exists an $f$ in $\mathcal{L}^\Psi(\mu)$ for which $\|f_n - f\|_\Psi \to 0$, by following these steps.

(i) Let $\{g_i\}$ be a nonnegative sequence in $\mathcal{L}^\Psi$ for which $\sum_{i\in\mathbb{N}} \|g_i\|_\Psi < C < \infty$. Define $G_n = \sum_{i\le n} g_i$ and $G = \sum_{i\in\mathbb{N}} g_i$. Show that $\mu\Psi(G_n/C) \le 1$ for every $n$. Justify a passage to the limit to deduce that $\mu\Psi(G/C) \le 1$ and $G(x) < \infty$ a.e.$[\mu]$.

(ii) Define $h_k = f_{n(k)}$ and $H_L := \sum_{k=L}^\infty |h_k - h_{k+1}|$. Use the results from part (i) to show that $\mu\Psi(H_L/\delta_{L-1}) \le 1$.

(iii) Show that $\{h_k(x)\}$ is a Cauchy sequence of real numbers for each $x$ at which $H_1(x)$ is finite. Deduce that

$$f(x) = \lim_{k\to\infty} h_k(x)\{H_1(x) < \infty\}$$

is a well defined real-valued function for which

$$H_L(x) \geq |h_L(x) - h_i(x)| \to |h_L(x) - h(x)| \qquad \text{as } i \to \infty.$$

(iv) Show that $\|h_L - h\|_\Psi \leq 2\delta_L$ and $h \in \mathcal{L}^\Psi$. Then explain why $\|f_n - h\|_\Psi \to 0$ as $n \to \infty$.