Statistics 330b/600b, Math 330b spring 2015
Homework # 6
Due: Thursday 26 February

*[1] Let μ and γ be finite measures defined on $(\mathcal{X}, \mathcal{A})$, with densities m and g with respect to some measure λ. Write $\mathcal{T}$ for the set of all $\mathcal{A}$-measurable functions ψ that take values in $[0, 1]$.

(i) Define $H(t) = \mu\{x : g(x) \geq tm(x)\}$, for $t \geq 0$. Show that H is decreasing and left-continuous, with $H(0) = \mu\mathcal{X}$ and $H(t) \to 0$ as $t \to \infty$.

(ii) For each constant α with $\mu\mathcal{X} > \alpha > 0$ explain why $\tau = \sup\{t : H(t) \geq \alpha\}$ is the largest τ for which $H(\tau) \geq \alpha$. Explain why there exists an $h \in \mathcal{T}$ (not necessarily unique) for which the function

$$\psi_0(x) = \{x : g(x) > \tau m(x)\} + h(x)\{x : g(x) = \tau m(x)\}$$

belongs to $\mathcal{T}$ and has $\mu\psi_0 = \alpha$.

(iii) For all $\psi \in \mathcal{T}$ show that $[\psi_0(x) - \psi(x)]\,[g(x) - \tau m(x)] \geq 0$ for all $x \in \mathcal{X}$. Deduce that $\gamma\psi_0 \geq \gamma\psi$ if $\mu\psi \leq \alpha$.

[2] Suppose X is a random variable taking values in $[0, \infty)$ for which $\mathbb{P}X = \mu < \infty$. Let $X_1, X_2, \dots$ be independent random variables, each with the same distribution as X. Define $Y_i := X_i\{X_i \leq i\}$ and $\mu_i := \mathbb{P}Y_i$. Let $S_n := \sum_{i \leq n} X_i$ and $T_n := \sum_{i \leq n} Y_i$.

For a fixed $\rho > 1$, let $\{k_n\}$ be an increasing sequence of positive integers such that $k_n/\rho^n \to 1$.

(i) Show that there exists a finite constant C for which $\sum_{j \in \mathbb{N}} \{i \leq k_j\}/k_j^2 \leq C/i^2$ for each positive integer i and $\sum_{i \geq \ell} i^{-2} \leq C/\ell$ for each positive integer ℓ.

(ii) Show that $\sum_{i \leq n} \{X > i\} \leq n \wedge X$. Deduce that

$$0 \leq \mu - \mathbb{P}T_n/n = \mathbb{P}\sum_{i \leq n} X\{X > i\}/n \to 0 \qquad \text{as } n \to \infty.$$

(iii) Show that $\sum_{i \in \mathbb{N}} \mathbb{P}\{X_i \neq Y_i\} \leq \sum_{i \in \mathbb{N}} \mathbb{P}\{X > i\} < \infty$. Deduce that $(S_n - T_n)/n \to 0$ almost surely. Hint: $n^{-1}\sum_{i \leq I} |X_i(\omega) - Y_i(\omega)| \to 0$ as $n \to \infty$, for each fixed I.

(iv) Show that $\text{var}(T_n) \leq \sum_{i \leq n} \mathbb{P}X^2\{X \leq i\}$.

(v) Use parts (i) and (iv) to show for each $\epsilon > 0$ that

$$\sum_{j \in \mathbb{N}} \mathbb{P}\{|T_{k_j} - \mathbb{P}T_{k_j}| > \epsilon k_j\} \leq \sum_{j \in \mathbb{N}} \sum_{i \in \mathbb{N}} \frac{\{i \leq k_j\}\mathbb{P}X^2\{X \leq i\}}{\epsilon^2 k_j^2}$$
$$\leq C\epsilon^{-2} \sum_{i \in \mathbb{N}} \mathbb{P}X^2\{X \leq i\}/i^2 < \infty.$$

Deduce that $(T_{k_j} - \mathbb{P}T_{k_j})/k_j \to 0$ almost surely as $j \to \infty$.

(vi) Deduce that $S_{k_j}/k_j \to \mu$ almost surely as $j \to \infty$.

(vii) For each $\rho' > \rho$, show that

$$\frac{S_{k_n}}{\rho' k_n} \leq \frac{S_m}{m} \leq \rho' \frac{S_{k_{n+1}}}{k_{n+1}} \qquad \text{for } k_n \leq m \leq k_{n+1},$$

when m is large enough.

(viii) Deduce that $\limsup S_m/m$ and $\liminf S_m/m$ both lie between μ/ρ' and $\mu\rho'$, with probability one.

(ix) Cast out a sequence of negligible sets as ρ decreases to 1 to deduce that $S_m/m \to \mu$ almost surely.

(x) Why does the SLLN for i.i.d. integrable random variables follow from the preceding argument?

[3] A set $\mathcal{H}$ of bounded, real-valued functions on a set $\mathcal{X}$ is called a λ-space if:

(a) It is a vector space under the operations of pointwise addition and pointwise multiplication by constants.

(b) The constant function $\mathbf{1}$ belongs to $\mathcal{H}$.

(c) If $\{h_n : n \in \mathbb{N}\} \subset \mathcal{H}$ and $h_n(x) \uparrow h(x)$ for each x and $\sup_x h(x) < \infty$ then $h \in \mathcal{H}$.

Define $\mathcal{A} := \{A \subseteq \mathcal{X} : \mathbf{1}_A \in \mathcal{H}\}$. Initially suppose also that $\mathcal{H}$ is stable under pairwise products of functions. Follow the first six steps to show that $\mathcal{A}$ is a sigma-field and $\mathcal{H}$ consists of precisely those bounded real functions that are $\mathcal{A}$-measurable.

(i) If $\{h_n : n \in \mathbb{N}\} \subset \mathcal{H}$ and $\sup_x |h_n(x) - h(x)| \to 0$, show that $h \in \mathcal{H}$. Hint: Take a subsequence $\{n(k) : k \in \mathbb{N}\}$ along which $\sup_x |h_{n(k)}(x) - h(x)| \le \delta_k := 2^{-k}$. Then show that

$$h_{n(k+1)}(x) + \delta_{k+1} + \delta_k \ge h(x) + \delta_k \ge h_{n(k)}(x).$$

Deduce that $h_{n(k)} + R_k$, for $R_k = 3\sum_{i\le k} \delta_i$, increases for to $h(x) + 3$, for each x.

(ii) Suppose $h \in \mathcal{H}$ and $\sup_x |h(x)| = M < \infty$. By a small variation on HW2.3, there exists polynomials p_n for which $\sup_{|t|\le M} |p_n(t) - t^+| \to 0$. Deduce that $h_n := p_n(h)$ converges uniformly to h, so that $h^+ \in \mathcal{H}$.

(iii) If $h_1, h_2 \in \mathcal{H}$ show that $h_1 \vee h_2$ and $h_1 \wedge h_2$ both belong to $\mathcal{H}$. Hint: $a \vee b = (a-b)^+ + b$ for all $a, b \in \mathbb{R}$.

(iv) Show that $\mathcal{A}$ is a sigma-field.

(v) Define $h_1 := \min(1, h^+)$ for some $h \in \mathcal{H}$. Show that $h_1 \in \mathcal{H}$ and that $\{h \ge 1\} = \lim_{n\to\infty} h_1^n$ belongs to $\mathcal{H}$. Deduce that h is $\mathcal{A}$-measurable. *Note: It is not enough just to have* $\{h \ge 1\} \in \mathcal{A}$.

(vi) Show that each bounded, $\mathcal{A}$-measurable function belongs to $\mathcal{H}$. Hint: Increasing limits of simple functions.

Now suppose $\mathcal{G}$ is a set of bounded real functions on $\mathcal{X}$ and $\mathcal{H}$ is the smallest λ-space for which $\mathcal{H} \supseteq \mathcal{G}$. Suppose also that $\mathcal{G}$ is stable under pairwise products.

(vii) Show that $\mathcal{H}_1 := \{h_1 \in \mathcal{H} : h_1 g \in \mathcal{H} \text{ for all } g \in \mathcal{G}\}$ is a λ-space. Deduce that $\mathcal{H}_1 = \mathcal{H}$.

(viii) Show that $\mathcal{H}_2 := \{h_2 \in \mathcal{H} : h_2 h \in \mathcal{H} \text{ for all } h \in \mathcal{H}\}$ is a λ-space. Deduce that $\mathcal{H}_2 = \mathcal{H}$, that is, $\mathcal{H}$ is stable under pairwise products.

(ix) Write $\sigma(\mathcal{G})$ for the smallest sigma-field on $\mathcal{X}$ for which each $g \in \mathcal{G}$ is $\sigma(\mathcal{G})\backslash\mathcal{B}(\mathbb{R})$-measurable. Show that $\sigma(\mathcal{G}) \subseteq \sigma(\mathcal{H}) = \mathcal{A}$. Deduce that every bounded, $\sigma(\mathcal{G})$-measurable real function belongs to $\mathcal{H}$.