Statistics 330b/600b, Math 330b spring 2015
Homework # 7
Due: Thursday 5 March

*[1] Suppose $f \in \mathcal{M}^+(\mathcal{X}, \mathcal{A})$. Show that $\{(x,t) \in \mathcal{X} \times \mathbb{R} : f(x) > t\}$ is $\mathcal{A} \otimes \mathcal{B}(\mathbb{R})$-measurable.

*[2] Let $\{\mu_i : i \in I\}$ be a countable set of finite measures defined on $(\mathcal{X}, \mathcal{A})$. Define $\mu : \mathcal{M}^+(\mathcal{X}, \mathcal{A}) \to [0, \infty]$ by $\mu g = \sum_{i \in I} \mu_i g$

(i) Show that μ corresponds to a countably additive measure on $\mathcal{A}$.

(ii) Extend Tonelli's theorem to measures expressible as countable sums of finite measures.

*[3] Let μ and ν be finite measures on $\mathcal{B}(\mathbb{R})$. Define distribution functions $F(t) := \mu(-\infty, t]$ and $G(t) := \nu(-\infty, t]$.

(i) Show that the sets $A_\mu = \{x \in \mathbb{R} : \mu\{x\} > 0\}$ and $A_\nu = \{x \in \mathbb{R} : \nu\{x\} > 0\}$ are both at most countable.

(ii) Define $D = \{(s,t) \in \mathbb{R}^2 : s = t\}$. Describe the function $h(t) = \nu^s\{(s,t) \in D\}$. Hint: Draw a suitably labelled picture.

(iii) Show that

$$\mu^t G(t) + \nu^t F(t) = \mu(\mathbb{R})\nu(\mathbb{R}) + \sum\nolimits_{x \in A_\mu \cap A_\nu} \mu\{x\}\nu\{x\},$$

Hint: Express the left-hand side as a $\mu \otimes \nu$ integral of a sum of two indicator functions.

(iv) Explain how (iii) is related to the integration-by-parts formula:

$$\int F(t) \frac{dG(t)}{dt}\, dt = F(\infty)G(\infty) - \int G(t) \frac{dF(t)}{dt}\, dt$$

[4] (Hard) Suppose X_1 and X_2 are random variables on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with $\|X_i\|_\Psi < \infty$ for some convex, increasing function Ψ on $\mathbb{R}^+$ with $\Psi(0) = 0$. Show that $\|X_1 - X_2\|_\Psi \geq \left\| \widetilde{X}_1 - \widetilde{X}_2 \right\|_\Psi$, where the $\widetilde{X}_i$ are constructed via the quantile transformation, as in UGMTP Example 4.32. Hint: Use the fact that $\Psi(x) = \int_0^x H(t)\, dt$, where H is an increasing right-continuous function on $\mathbb{R}^+$. Represent $H(t)$ as $\mu(0, t]$ for some measure μ. You need not reprove any results from UGMTP.