Statistics 330b/600b, Math 330b spring 2015
Homework # 8
Due: Thursday 26 March

*[1] Suppose T is a function from a set $\mathcal{X}$ into a set $\mathcal{Y}$, which is equipped with a σ-field $\mathcal{B}$. Recall that $\sigma(T) := \{T^{-1}B : B \in \mathcal{B}\}$ is the smallest sigma-field on $\mathcal{X}$ for which T is $\sigma(T)\backslash\mathcal{B}$-measurable.

Show that to each f in $\mathcal{M}^+(\mathcal{X}, \sigma(T))$ there exists a g in $\mathcal{M}^+(\mathcal{Y}, \mathcal{B})$ such that $f = g \circ T$ (that is, $f(x) = g(T(x))$, for all x in $\mathcal{X}$) by following these steps.

(i) If f is the indicator function of $T^{-1}(B)$ and g is the indicator function of B, show that $f = g \circ T$.

(ii) Extend to the case where $f \in \mathcal{M}^+_{\text{simple}}(\mathcal{X}, \sigma(T))$.

(iii) Suppose $f_n = g_n \circ T$ is a sequence in $\mathcal{M}^+_{\text{simple}}(\mathcal{X}, \sigma(T))$ that increases pointwise to f. Define $g(y) = \limsup g_n(y)$ for each y in $\mathcal{Y}$. Show that g has the desired property.

(iv) In part (iii), why can't you assume that $\lim g_n(y)$ exists for each y in $\mathcal{Y}$?

*[2] Suppose X and Y are independent real-valued random variables with

$$\mathbb{P}\{X = t\}\mathbb{P}\{Y = t\} = 0 \qquad \text{for each } t \in \mathbb{R}.$$

Show that $\mathbb{P}\{X = Y\} = 0$. Hint: Tonelli.

*[3] I class I considered a problem with P equal to Lebesgue measure on $\mathcal{B}(0,1)^2$ and T the map from $(0,1)^2 \to (0,1)$ with $T(x_1, x_2) = \max(x_1, x_2)$. I claimed that T has distribution Q with density $2t\{0 < t < 1\}$ with respect to Lebesgue measure and asserted that P has a disintegration $\{P_t : 0 < t < 1\}$, where P_t is the uniform distribution on the set $\{T = t\}$. Establish the validity of this assertion by checking that $P(0,a] \times (0,b] = Q^t P_t^x (0,a] \times (0,b]$ for all $a, b \in (0,1)$. You will then need some sort of generating class argument to complete the proof. [Please do not just copy the argument from UGMTP.]

[4] Define $g(x) = \sum_{i \le n} |x_i|^p$ where $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ and $p > 0$. Show that the set $\{x \in \mathbb{R}^n : g(x) \le 1\}$ has volume (Lebesgue measure)

$$V = \frac{(2\Gamma(1 + 1/p))^n}{\Gamma(1 + n/p)}$$

where $\Gamma(\alpha) = \int_0^\infty t^{\alpha-1}e^{-t}dt$ for $\alpha > 0$. Argue as follows. (For an extra challenge, replace g by $\sum_{i \le n} |x_i|^{p_i}$ for constants $p_i > 0$ then find the formula for the volume.) Define

$$J = \int_{\mathbb{R}^n} e^{-g(x)}\, dx.$$

(i) Factorize J into a product of n integrals over $\mathbb{R}$ then evaluate using the Calculus that you know.

(ii) Show that $J = \int_{\mathbb{R}^n} \int_0^\infty e^{-t}\{t \ge g(x)\}\, dt\, dx$ then use Tonelli.

(iii) Then what?