Statistics 330b/600b, Math 330b spring 2015
Homework # 9
Due: Thursday 2 April

*[1] Write $\mathcal{G}$ for the set of all open subsets of a topological space $\mathcal{X}$. Suppose the topology is countably generated, that is, there exists a countable $\mathcal{G}_0 \subset \mathcal{G}$ such that $G = \bigcup\{G_0 : G \supseteq G_0 \in \mathcal{G}_0\}$ for each $G \in \mathcal{G}$. Show that such a $\mathcal{B}(\mathcal{X}) = \sigma(\mathcal{G}_0)$.

*[2] Let $\mathcal{X}$ and $\mathcal{Y}$ be topological spaces equipped with their Borel sigma-fields $\mathcal{B}(\mathcal{X})$ and $\mathcal{B}(\mathcal{Y})$. Equip $\mathcal{X}\times\mathcal{Y}$ with the product topology and its Borel sigma-field $\mathcal{B}(\mathcal{X}\times\mathcal{Y})$. (The open sets in the product space are, by definition, all possible unions of sets $G \times H$, with G open in $\mathcal{X}$ and H open in $\mathcal{Y}$.)

(i) Show that $\mathcal{B}(\mathcal{X}) \otimes \mathcal{B}(\mathcal{Y}) \subseteq \mathcal{B}(\mathcal{X} \times \mathcal{Y})$.

(ii) If both $\mathcal{X}$ and $\mathcal{Y}$ have countably generated topologies, prove that $\mathcal{B}(\mathcal{X}) \otimes \mathcal{B}(\mathcal{Y}) = \mathcal{B}(\mathcal{X} \times \mathcal{Y})$.

(iii) Explain why $\mathcal{B}(\mathbb{R}^n) = \mathcal{B}(\mathbb{R}^k) \otimes \mathcal{B}(\mathbb{R}^{n-k})$.

*[3] Suppose $(\mathcal{X}, \mathcal{A}, \mathbb{P})$ is a probability space and $\mathcal{B}$ is a sub-sigma-field of $\mathcal{A}$. UGMTP Problem 5.28 almost asserts: if $g \in \mathcal{L}^2(\mathcal{X}, \mathcal{A}, \mathbb{P})$ then the conditional expectation $\mathbb{P}_{\mathcal{B}}(g)$ is (almost surely) the orthogonal projection of g onto the subspace $\mathcal{L}^2(\mathcal{X}, \mathcal{B}, \mathbb{P})$. Give a complete, rigorous proof of this assertion, starting from properties known for $\mathbb{P}_{\mathcal{B}}$ as a map (defined up to almost sure equivalence) from $\mathcal{M}^+(\mathcal{X}, \mathcal{A})$ into $\mathcal{M}^+(\mathcal{X}, \mathcal{B})$.

The next problem is only for the very brave.

[4] Let P be a probability measure on $\mathcal{B}(\mathbb{R})$, not necessarily dominated by Lebesgue measure. (In particular, it might have atoms.) Define $\mathbb{P} = P \otimes P$, the product measure on $\mathcal{B}(\mathbb{R}^2)$. Let $T(x, y) = \min(x, y)$ and $Q = T\mathbb{P}$. For each $t \in \mathbb{R}$ define

$$\begin{aligned}
\alpha(t) &= P\{t\} \\
\pi_0(t) &= P(t, \infty) \qquad \text{AND} \qquad \pi_1(t) = P[t, \infty) \\
q(t) &= \pi_0(t) + \pi_1(t) = \alpha(t) + 2\pi_0(t).
\end{aligned}$$

and

$$\gamma(t) = \begin{cases} \alpha(t)/q(t) & \text{if } \alpha(t) > 0 \\ 0 & \text{if } \alpha(t) = 0 \end{cases} \qquad \text{and } \beta(t) = \tfrac{1}{2}(1 - \gamma(t)).$$

For t such that $\pi_0(t) > 0$ define ν_t by $d\nu_t/dP = g_t(x) = \{x > t\}/\pi_0(t)$. (You should decide whether it is necessary to define ν_t when $\pi_0(t) = 0$.)

(i) Show that $dQ/dP = q(t)$. (More precisely, q is one version of the density, which is defined up to a P-equivalence.)

(ii) Show that the conditional distribution $\mathbb{P}(\cdot \mid T = t)$ can be defined (Q-almost everywhere) as

$$\gamma(t)\left(\delta_t \otimes \delta_t\right) + \beta(t)\left(\delta_t \otimes \nu_t\right) + \beta(t)\left(\nu_t \otimes \delta_t\right)$$

where δ_t denotes the point mass at t. Hint: I found it helpful to split $P^x P^y f(x, y)$ into contributions from the sets $\{x = y\}$ and $\{x > y\}$ and $\{x < y\}$.