Statistics 330b/600b, Math 330b spring 2016
Homework # 10
Due: Thursday 14 April

*[1] Suppose $\{(X_i, \mathcal{F}_i) : i \in \mathbb{N}_0\}$ is a martingale (on $\Omega, \mathcal{F}, \mathbb{P}$) with $\sup_i \mathbb{P}X_i^2 < \infty$. By the following steps show that X_i converges almost surely and in $\mathcal{L}^2$ to some random variable X_∞ in $\mathcal{L}^2(\Omega, \mathcal{F}_\infty, \mathbb{P})$, where $\mathcal{F}_\infty := \sigma(\mathcal{E})$ with $\mathcal{E} := \cup_{i \in \mathbb{N}_0} \mathcal{F}_i$.

(i) Write X_n as $X_0 + \sum_{i=1}^n \xi_i$ where $\mathbb{P}(\xi_i \mid \mathcal{F}_{i-1}) = 0$ almost surely and $\mathbb{P}\xi_i^2 = \sigma_i^2$. Show that $\mathbb{P}X_n^2 = \mathbb{P}X_0^2 + \sum_{i=1}^n \sigma_i^2$. Hint: What do you know about $\mathbb{P}(\xi_i \xi_j \mid \mathcal{F}_i)$ if $i < j$?

(ii) For $n < m$ show that $\mathbb{P}|X_n - X_m|^2 \leq \sum_{i \geq n} \sigma_i^2 \to 0$ as $n \to \infty$. Via completeness of $\mathcal{L}^2(\Omega, \mathcal{F}_\infty, \mathbb{P})$ deduce existence of an X_∞ for which $\mathbb{P}|X_n - X_\infty|^2 \to 0$ as $n \to \infty$.

(iii) For $i < n$ and $F \in F_i$ show that $\mathbb{P}X_i F = \mathbb{P}X_n F \to \mathbb{P}X_\infty F$ as $n \to \infty$. Deduce that $X_i = \mathbb{P}(X_\infty \mid \mathcal{F}_i)$ almost surely. Hint: Cauchy-Schwarz.

(iv) Define $Y_n = \mathbb{P}(X_\infty^+ \mid \mathcal{F}_n)$. Use facts about positive (super)martingales to deduce that Y_n converges almost surely and in $\mathcal{L}^1$ to some Y_∞ in $\mathcal{L}^1(\Omega, \mathcal{F}_\infty, \mathbb{P})$.

(v) Argue similarly that $Z_n := \mathbb{P}(X_\infty^- \mid \mathcal{F}_n) \to Z_\infty$ almost surely and in $\mathcal{L}^1$. Deduce that $X_n \to W := Y_\infty - Z_\infty$ almost surely and in $\mathcal{L}^1$.

(vi) From the facts that $X_n \to X_\infty$ in $\mathcal{L}^2$ and $X_n \to W$ in $\mathcal{L}^1$, deduce that $X_\infty = W$ almost surely. Hint: First explain why $\mathbb{P}X_\infty F = \mathbb{P}WF$ for all $F \in \mathcal{E}$. Then what?

*[2] Consider again the branching process $Z_{n+1} = \sum_{j \in \mathbb{N}} \{j \leq Z_n\} \xi_{n+1,j}$ discussed in class. Remember that all the ξ_{ij}'s are independent, each with the same distribution as a ξ for which $\mathbb{P}\{\xi = k\} = p_k$ for $k \in \{0\} \cup \mathbb{N}$. Remember that we assume $p_0 > 0$ to avoid a trivial case.

(i) For iid random variables $\xi_1, \ldots, \xi_k$, each distributed like ξ, show that $\mathbb{P}\{\xi_1 + \cdots + \xi_k = k\} \leq 1 - p_0^k$ for each $k \in \mathbb{N}$.

(ii) For $k, n \in \mathbb{N}$ define $A_{n,k} = \{Z_i = k \text{ for } i \geq n\}$. Prove that $\mathbb{P}A_{n,k} = 0$. Hint: Bound $\mathbb{P}\{X_i = k \text{ for } m \geq i \geq n\}$ by conditioning.

(iii) Deduce that $\mathbb{P}\{\omega : Z_i(\omega) \to k\} = 0$ for each $k \in \mathbb{N}$.

[3] Suppose an urn initially contains r_0 red balls and $b_0 = N_0 - r_0$ black balls. At step n (for $n \in \mathbb{N}$) a ball is selected at random from the urn then thrown back together with another d_n balls of the same color. Define $\xi_n := \{\text{red ball selected at } n\text{th step}\}$ and $\mathcal{F}_n := \sigma\{\xi_i : i \leq n\}$. The nonnegative number d_n is random and $\mathcal{F}_{n-1}$-measurable.

After n steps of this procedure the urn contains $N_n := N_0 + \sum_{i \leq n} d_i$ balls of which $R_n = r_0 + \sum_{i \leq n} \xi_i d_i$ are red. The proportion of red balls in the urn is $X_n = R_n / N_n$.

Show that $\{(X_n, \mathcal{F}_n) : n \in \mathbb{N}_0\}$ is a martingale. You will need to define $\mathcal{F}_0$ and X_0 suitably.