Statistics 330b/600b, Math 330b spring 2016
Homework # 12
Due: Thursday 21 April

*[1] Suppose A is a subset of a metric space with interior $\mathring{A}$. Show that the indicator function of $\mathring{A}$ equals $\mathring{f}$, the largest lower semi-continuous function for which $A \geq \mathring{f}$.

*[2] Suppose $(\mathcal{X}, d)$ is a metric space with a countable, dense subset $\{x_i : i \in \mathbb{N}\}$. Write $\mathcal{P}(\mathcal{X})$ for the set of all probability measures on $\mathcal{B}(\mathcal{X})$. For $P, Q \in \mathcal{P}(\mathcal{X})$ define

$$D(P,Q) = \sup\{|Pf - Qf| : \|f\|_{BL} \leq 1\}.$$

(i) Show that D is a metric on $\mathcal{P}(\mathcal{X})$.

(ii) For a given $\epsilon > 0$ define $h_0(x) \equiv \epsilon$ and $h_i(x) = (1 - d(x, x_i)/\epsilon)^+$ for $i \geq 1$. Define $H_k(x) = \sum_{i=0}^k h_i(x)$. Show that $\{H_k(x) \leq 1/2\} \downarrow \emptyset$ as $k \uparrow \infty$. Hint: What do you know about $H_k(x)$ if $k \geq i$ and $d(x, x_i) < \epsilon/2$?

(iii) Define $\ell_{i,k} = h_i/H_k$ for $0 \leq i \leq k$. Show that each $\ell_{i,k}$ belongs to $\mathrm{BL}(\mathcal{X})$ and $\sum_{i=0}^k \ell_{i,k}(x) = 1$ for every x. Hint: First show that $1/H_k \in \mathrm{BL}(\mathcal{X})$.

(iv) For each f with $\|f\|_{BL} \leq 1$ show that

$$|f(x) - \sum\nolimits_{i=1}^k f(x_i)\ell_{i,k}(x)| \leq \epsilon + f(x)\ell_{0,k}(x) \leq 3\epsilon + \{H_k(x) \leq 1/2\}.$$

(v) Suppose P and $\{P_n : n \in \mathbb{N}\}$ are probability measures on $\mathcal{B}(\mathcal{X})$ for which $P_n f \to Pf$ for each f in $\mathrm{BL}(\mathcal{X})$. Show that $D(P_n, P) \to 0$. You may assume that $\limsup_n P_n F \leq PF$ for each closed set F. (The proof is almost the same as a result proved in class.)

[3] Suppose W_n has a Poisson(n) distribution. Show that $Z_n := \sqrt{W_n} - \sqrt{n} \rightsquigarrow N(0, 1/4)$ by following these steps. (Alternatively, you could also use UGMTP Example 7.31.) If you want more of a challenge, replace n by a real number λ that goes off to infinity.

(i) Show that $X_n := (W_n - n)/\sqrt{n} \rightsquigarrow N(0,1)$. (You may assume without proof that the sum of two independent Poissons is also Poisson.)

(ii) Show that $Y_n := \sqrt{W_n/n} \to 1$ in probability. That is, $\mathbb{P}\{|Y_n - 1| > \epsilon\} \to 0$ for each fixed $\epsilon > 0$.

(iii) Show that $Z_n = T(X_n, Y_n)$ for some $T : \mathbb{R}^2 \to \mathbb{R}$ that is continous at the right places. Deduce that $Z_n \rightsquigarrow N(0, 1/4)$.