Statistics 330b/600b, Math 330b spring 2016
Homework # 2
Due: Thursday 4 February

Please attempt at least the starred problems. Please explain your reasoning.

*[1] A set $\mathcal{E}$ of subsets of $\mathcal{X}$ is called a field if: it contains $\emptyset$ and is stable under complements, finite unions, and finite intersections. Suppose μ is a finite measure (that is, $\mu\mathcal{X} < \infty$) on a sigma-field $\mathcal{A} = \sigma(\mathcal{E})$, where $\mathcal{E}$ is a field of subsets of $\mathcal{X}$. In class I began the argument to show that every A in $\mathcal{A}$ has the following property:

$$(\star)\text{: for each } \epsilon > 0 \text{ there exists an } E \in \mathcal{E} \text{ for which } \mu(A\Delta E) < \epsilon.$$

Remember that $A\Delta E := (A \cap E^c) \cup (A^c \cap E)$.

Please write out a complete proof of this result. Personally I would shrink from attempting a proof using only Boolean algebra; the calculations with indicator functions are far cleaner. Nevertheless, you might learn something from the pain inflicted by the Boolean constraints. Your choice.

Remember that the proof can be carried out by showing that

$$\mathcal{A}_0 := \{A \in \mathcal{A} : A \text{ has property } (\star)\}$$

is a sigma-field for which $\mathcal{A}_0 \supseteq \mathcal{E}$.

*[2] Suppose $f_1, \dots, f_k \in \mathcal{M}^+(\mathcal{X}, \mathcal{A})$ and $\theta_1, \dots, \theta_k$ are strictly positive numbers that sum to one. Let μ be a measure on $\mathcal{A}$. Show that

$$\mu \prod\nolimits_{i\le k} f_i^{\theta_i} \le \prod\nolimits_{i\le k} (\mu f_i)^{\theta_i}$$

by following these steps. You may assume that $\mu f_i < \infty$ for all i, for otherwise the assertion is trivially true (provided all the other μf_j are strictly positive).

(i) Explain why the inequality is trivially true (because the left-hand side is zero) if $\mu f_i = 0$ for at least one i.

(ii) Explain why there is no loss of generality in assuming that $\mu f_i = 1$ for each i and $f_i(x) < \infty$ for each x and i.

(iii) For all $a_1, \dots, a_k \in \mathbb{R}^+$, show that $\prod_{i\le k} a_i^{\theta_i} \le \sum_{i\le k} \theta_i a_i$. Hint: First dispose of the trivial case where at least one a_i is zero, then rewrite the inequality using $b_i = \log a_i$. *You do not need to reprove that the* log *function is concave on* $(0, \infty)$. If you do not exclude the case where $\min_i a_i = 0$ be prepared to explain what concavity means for a function that can take the value $-\infty$.

(iv) Complete the proof by considering the inequality from (iii) with $a_i = f_i(x)$.

Remark. Textbooks often contain the the special case where $k = 2$ and $\theta_1 = 1/p$ and $\theta_2 = 1/q$ and $f_1 = |g_1|^p$ and $f_2 = |g_2|^q$, with the assertion that $|\mu(g_1 g_2)| \le \mu|g_1 g_2| \le (\mu|g_1|^p)^{1/p} (\mu|g_2|^q)^{1/q}$.

[3] Let $A_1, \dots, A_N$ be events in a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. For each subset J of $\{1, 2, \dots, N\}$ write A_J for $\cap_{i\in J} A_i$. Define $S_k := \sum_{|J|=k} \mathbb{P}A_J$, where $|J|$ denotes the number of indices in J. For $0 \le m \le n$ define

$$B_m = \{\text{exactly } m \text{ of the } A_i\text{'s occur}\} = \{\omega \in \Omega : \sum\nolimits_{i=1}^N \mathbf{1}_{A_i}(\omega) = m\}.$$

(i) Explain why

$$B_m = \sum\nolimits_{|J|=m} \prod\nolimits_{i \in J} A_i \prod\nolimits_{j \in J^c} (1 - A_j).$$

Remember I am writing A_i instead of $\mathbf{1}_{A_i}$.

(ii) Deduce that

$$B_m = \sum\nolimits_{\ell=m}^{N} (-1)^{\ell-m} \sum\nolimits_{|K|=\ell} \binom{\ell}{m} A_K.$$

(iii) Take expectations (integrals with respect to $\mathbb{P}$) to deduce that

$$\mathbb{P}B_m = \binom{m}{m} S_m - \binom{m+1}{m} S_{m+1} + \cdots \pm \binom{N}{m} S_N.$$

Compare with the method suggested in UGMTP Problem 1.1. (You may use that method if you prefer it.)