**Statistics 330b/600b, Math 330b spring 2016**
Homework # 3
Due: Thursday 11 February

*Please attempt at least the starred problems. Please explain your reasoning.*

*[1] Suppose $\mathcal{A}$ is a sigma-field on a set $\mathcal{X}$ and $\mu$ is a measure on $\mathcal{A}$. Write $\mathcal{N}_\mu$ for $\{N \in \mathcal{A} : \mu N = 0\}$. Define

$$\mathcal{A}_\mu := \{B \subseteq \mathcal{X} : \exists A \in \mathcal{A},\ N \in \mathcal{N}_\mu \text{ such that } |B - A| \le N\}.$$

(i) Show that $\mathcal{A}_\mu$ is a sigma-field. Hint: If $\{B_i : i \in \mathbb{N}\} \subseteq \mathcal{A}_\mu$ and $|B_i - A_i| \le N_i$, show that $|\cup_i B_i - \cup_i A_i| \le \cup_i N_i$.

(ii) If $|B - A_i| \le N_i$ for $i = 1, 2$, with $A_i \in \mathcal{A}$ and $N_i \in \mathcal{N}_\mu$, show that $\mu A_1 = \mu A_2$. Hint: Show $A_1 \le A_2 + N_1 + N_2$.

(iii) If $|B - A| \le N$ with $A \in \mathcal{A}$ and $N \in \mathcal{N}_\mu$ define $\nu B = \mu A$. Show that $\nu$ is a measure on $\mathcal{A}_\mu$ whose restriction to $\mathcal{A}$ equals $\mu$.

*[2] For $f$ in $\mathcal{L}^1 := \mathcal{L}^1(\mathcal{X}, \mathcal{A}, \mu)$ define $\|f\|_1 = \mu|f|$. Let $\{f_n\}$ be a Cauchy sequence in $\mathcal{L}^1$, that is, $\|f_n - f_m\|_1 \to 0$ as $\min(m, n) \to \infty$. Show that there exists an $f$ in $\mathcal{L}^1$ for which $\|f_n - f\|_1 \to 0$, by following these steps. Note: Don't confuse Cauchy sequences (in $\mathcal{L}^1$ distance) of functions with Cauchy sequences of real numbers.

(i) For each $k \in \mathbb{N}$ there exists an $n(k) \in \mathbb{N}$ for which: $\|f_n - f_m\|_1 < 2^{-k}$ when $\min(m, n) \ge n(k)$. Without loss of generality assume that $n(k)$ is strictly increasing with $k$. Define $H(x) := \sum_{k=1}^{\infty} |f_{n(k)}(x) - f_{n(k+1)}(x)|$. Show that $\mu H < \infty$.

(ii) Define $g_k = f_{n(k)}$. Show that $\{g_k(x) : k \in \mathbb{N}\}$ is a Cauchy sequence of real numbers for each $x$ in the set $\{H < \infty\}$. Explain why

$$f(x) = \liminf_{k \to \infty} g_k(x)\{H(x) < \infty\}$$

is a real-valued, $\mathcal{A}$-measurable function for which

$$H(x) \ge |g_k(x) - f(x)| \to 0 \qquad \text{a.e. } [\mu].$$

(iii) Deduce that $\|g_k - f\|_1 \to 0$ as $k \to \infty$.

(iv) Show that $f$ belongs to $\mathcal{L}^1(\mu)$ and $\|f_n - f\|_1 \to 0$ as $n \to \infty$.

[3] Suppose $\{X_n : n \in \mathbb{N}\}$ is a sequence of real-valued random variables on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. (That is, each $X_n$ is an $\mathcal{A}\backslash\mathcal{B}(\mathbb{R})$-measurable function.) Suppose also that the sequence is Cauchy in probability, that is, for each $\epsilon > 0$ there exists an $n_\epsilon$ for which $\mathbb{P}\{|X_n - X_m| > \epsilon\} < \epsilon$ for all $m, n \ge n_\epsilon$.

(i) Write $n(k)$ for the $n_\epsilon$ corresponding to $\epsilon = 2^{-k}$, for $k \in \mathbb{N}$. [You may assume that $n(1) < n(2) < \dots$. Why?] Show that

$$\sum\nolimits_{k \in \mathbb{N}} \mathbb{P}\{|X_{n(k)} - X_{n(k+1)}| > 2^{-k}\} < \infty.$$

(ii) Deduce that there exists a $\mathbb{P}$-negligible set $N$ for which $\{X_{n(k)}(\omega) : k \in \mathbb{N}\}$ is a Cauchy sequence of real numbers for each $\omega$ in $N^c$. Deduce that $X_{n(k)}$ converges almost surely to the real-valued random variable $X(\omega) := \limsup_k \{\omega \in N^c\} X_{n(k)}(\omega)$.

(iii) Use Dominated Convergence to show that $\mathbb{P}\{|X_{n(k)} - X| > \epsilon\} \to 0$ as $k \to \infty$, for each fixed $\epsilon > 0$.

(iv) Deduce that $\mathbb{P}\{|X_n - X| > \epsilon\} \to 0$ as $n \to \infty$, for each fixed $\epsilon > 0$.

[4] Let $A_1, A_2, \dots$ be events in a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Define $X_n = A_1 + \cdots + A_n$ and $\sigma_n = \mathbb{P}X_n$.

(i) Show that $\|X_n/\sigma_n\|_2 \geq 1$. Hint: Jensen.

(ii) Show that (as a pointwise inequality between random variables)

$$\{X_n = 0\} \leq \frac{(k - X_n)(k + 1 - X_n)}{k(k+1)}$$

for each positive integer $k$. Hint: Are there any values of $X_n$ for which the ratio on the right-hand side is negative?

*Now suppose* $\sigma_n \to \infty$ *and* $\|X_n/\sigma_n\|_2 \to 1$.

(iii) By making an appropriate choice of the integer $k$ (depending on $n$) in (ii), show that $\mathbb{P}\{X_n = 0\} \to 0$ as $n \to \infty$. Deduce that $\sum_1^\infty A_i \geq 1$ almost surely.

(iv) Prove that $\sum_{i=m}^\infty A_i \geq 1$ almost surely, for each fixed $m$. Hint: Show that the two convergence assumptions also hold for the sequence $A_m, A_{m+1}, \dots$.

(v) Deduce that $\mathbb{P}\{\omega \in A_i \text{ i. o. }\} = 1$.

(vi) Suppose $\{B_i\}$ is a sequence of events for which $\sum_i \mathbb{P}B_i = \infty$ and $\mathbb{P}(B_i B_j) = (\mathbb{P}B_i)(\mathbb{P}B_j)$ for all $i \neq j$. Show that $\mathbb{P}\{\omega \in B_i \text{ i. o. }\} = 1$. *Please use (v). I am not interested in seeing the standard textbook proof for the harder direction of Borel-Cantelli.*