Statistics 330b/600b, Math 330b spring 2016
Homework # 4
Due: Thursday 18 February

Please attempt at least the starred problems. Please explain your reasoning.

*[1] Suppose $(\mathcal{X}, \mathcal{A}, \mu)$ is a measure space and $f \in \mathcal{L}^1(\mathcal{X}, \mathcal{A}, \mu)$. Show that, for each $\epsilon > 0$ there exists a $\delta > 0$ for which $|\mu(FA)| < \epsilon$ for every set A in $\mathcal{A}$ for which $\mu A < \delta$. Hint: Show that there is a constant M for which $\mu|f|\{|F| > M\} < \epsilon/2$. Then bound $|\mu(fA)|$ by the sum of contribution from $\{|f| \leq M\}$ and its complement.

*[2] Let P and Q be finite measures defined on the same sigma-field $\mathcal{F}$, with densities p and q with respect to a measure μ. Suppose $\mathcal{X}_0$ is a measurable set with the property that there exists a nonnegative constant K such that $q \geq Kp$ on $\mathcal{X}_0$ and $q \leq Kp$ on $\mathcal{X}_0^c$. For each $\mathcal{F}$-measurable function with $0 \leq f \leq 1$ and $Pf \leq P\mathcal{X}_0$, prove that $Qf \leq Q\mathcal{X}_0$. Hint: Prove that $(q-Kp)(2\mathcal{X}_0-1) = |q-Kp| \geq (q-Kp)(2f-1)$, then integrate. *To statisticians this result is known as the Neyman-Pearson Lemma. The function f defines a randomized test between P and Q: reject P with probability $f(x)$ if x is observed.*

[3] Suppose $\mathcal{A}$ is a sigma-field on a set $\mathcal{X}$ and P and Q are probability measures on $\mathcal{A}$. Suppose also that P has density p and Q has density q with respect to some measure μ.

(i) For each $A \in \mathcal{A}$ show that $\sqrt{(PA)(QA)} \geq \mu\left(\sqrt{pq}A\right)$.

(ii) For each partition $\mathcal{X} = A_1 + \cdots + A_k$ of $\mathcal{X}$ into $\mathcal{A}$-measurable sets, show that

$$\mu\left(\sqrt{p} - \sqrt{q}\right)^2 \geq \sum\nolimits_i \left(\sqrt{PA_i} - \sqrt{QA_i}\right)^2.$$

(iii) Show that the supremum of $\sum_i \left(\sqrt{PA_i} - \sqrt{QA_i}\right)^2$ over all finite partitions of $\mathcal{X}$ is equal to $\mu\left(\sqrt{p} - \sqrt{q}\right)^2$. Hint: Explain why, without loss of generality, $\mu = P + Q$. Then consider sets $A_0 = \{p = 0\}$ and $A_i = \{a_{i-1} < p \leq a_i\}$ where $0 = a_0 < a_1 < \cdots < a_k$ and $\max_i(a_i - a_{i-1}) < \epsilon$.