Statistics 330b/600b, Math 330b spring 2016
Homework # 5
Due: Thursday 25 February

Please attempt at least the starred problems. Please explain your reasoning.

*[1] Suppose μ and ν are finite measures on $(\mathcal{X}, \mathcal{A})$, where $\mathcal{A} = \sigma(\mathcal{E})$ and $\mathcal{E}$ is a field. Follow these steps to show that $\nu \ll \mu$ (that is, if $A \in \mathcal{A}$ and $\mu A = 0$ then $\nu A = 0$) if and only if: $(\star)$ for each $\epsilon > 0$ there exists a $\delta > 0$ such that $\nu E < \epsilon$ for each $E \in \mathcal{E}$ with $\mu E < \delta$.

(i) If $(\star)$ fails, explain why there must exist some $\epsilon > 0$ and sets $E_k \in \mathcal{E}$ such that $\mu E_k < 2^{-k}$ and $\nu E_k \geq \epsilon$, for $k \in \mathbb{N}$. Define $A_k = \cup_{i \geq k} E_i$ and $A = \cap_{k \in \mathbb{N}} A_k$. Show that $\mu A = 0$ but $\nu A \geq \epsilon$.

(ii) Define $\lambda = \mu + \nu$. Suppose $(\star)$ holds and $\mu A = 0$ for some $A \in \mathcal{A}$. Show that $\nu A = 0$. Hint: Use HW Problem 2.1 to approximate A in $\mathcal{L}^1(\lambda)$ by a set from $\mathcal{E}$.

*[2] Suppose $(\mathcal{X}, \mathcal{A}, \mu)$ is a measure space. Show that μ is sigma-finite if and only if there exists an $\mathcal{A}$-measurable function for which $\mu f < \infty$ and $f(x) > 0$ for all $x \in \mathcal{X}$.

*[3] Suppose $f_1, f_2 \in \mathcal{L}^1(\mathcal{X}, \mathcal{A}, \mu)$. Show that

$$\sqrt{(\mu f_1)^2 + (\mu f_2)^2} \leq \mu \sqrt{f_1^2 + f_2^2}.$$

(Later in the course $f(x) = f_1(x) + i f_2(x)$ will be a complex-valued function, with $\mu f := \mu f_1 + i \mu f_2$. The inequality then becomes $|\mu f| \leq \mu |f|$, with $|\cdot|$ denoting the modulus of a complex number.) Hint: Define $F = \sqrt{f_1^2 + f_2^2}$. Explain why, without loss of generality, we may assume $\mu F = 1$. Define a probability measure P by $dP/d\mu = F$. Reduce the asserted inequality to an analogous inequality involving P and the functions $g_i := f_i\{F > 0\}/F$.

[4] Let P be a probability measure on $\mathcal{B}(\mathbb{R})$. Define

$$m_0 := \inf\{t : P(-\infty, t] \geq 1/2\}.$$

Show that $P[m_0, \infty) \geq 1/2$ and $P(-\infty, m_0] \geq 1/2$. *[The value m_0 is called a median for P.]*

[5] Suppose $Z = X + Y$, with X and Y independent random variables. Let m be a median for the distribution of Y. Show that $\mathbb{P}\{X \geq x\} \leq 2\mathbb{P}\{Z \geq x + m\}$ for each real x.