Statistics 330b/600b, Math 330b spring 2016
Homework # 6
Due: Thursday 3 March

Please attempt at least the starred problems. Please explain your reasoning.

*[1] Suppose $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space and $T_i : \Omega \to \mathcal{X}_i$ is an $\mathcal{F}\backslash\mathcal{B}_i$-measurable function, for some sigma-field $\mathcal{B}_i$ on the set $\mathcal{X}_i$, for $i = 1, 2$. Suppose also that $\mathcal{B}_i = \sigma(\mathcal{D}_i)$, where $\mathcal{D}_i$ is stable under pairwise intersections.

(i) Show that $\mathcal{F}_i := \{T_i^{-1}B_i : B_i \in \mathcal{B}_i\}$ is the smallest sigma-field on ω for which T_i is $\mathcal{F}_i\backslash\mathcal{B}_i$-measurable. *[The sigma-field $\mathcal{F}_i$ is often denoted by $\sigma(T_i)$.]*

(ii) Show that T_1 and T_2 are independent if

$$\mathbb{P}\{T_1 \in D_1, T_2 \in D_2\} = \mathbb{P}\{T_1 \in D_1\}\mathbb{P}\{T_2 \in D_2\} \qquad \text{for all } D_i \in \mathcal{D}_i.$$

*[2] Suppose $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space and $X_i : \Omega \to \mathcal{X}_i$ is an $\mathcal{F}\backslash\mathcal{B}_i$-measurable function, for some sigma-field $\mathcal{B}_i$ on the set $\mathcal{X}_i$, for $1 \le i \le 5$. Define functions $T : \Omega \to \mathcal{X}_1 \times \mathcal{X}_2$ and $S : \mathcal{X}_3 \times \mathcal{X}_4 \times \mathcal{X}_5$ by

$$T(\omega) := (X_1(\omega), X_2(\omega)) \quad \text{AND} \quad S(\omega) := (X_3(\omega), X_4(\omega), X_5(\omega))$$

(i) Equip $\mathcal{X}_1 \times \mathcal{X}_2$ with its product sigma-field

$$\mathcal{B}_1 \otimes \mathcal{B}_2 := \sigma\{B_1 \times B_2 : B_i \in \mathcal{B}_i \text{ for } i = 1, 2\}.$$

Equip $\mathcal{X}_3 \times \mathcal{X}_4 \times \mathcal{X}_5$ with the analogously defined $\mathcal{B}_3 \otimes \mathcal{B}_4 \otimes \mathcal{B}_5$. Show that T is $\mathcal{F}\backslash(\mathcal{B}_1 \otimes \mathcal{B}_2)$-measurable and S is $\mathcal{F}\backslash(\mathcal{B}_3 \otimes \mathcal{B}_4 \otimes \mathcal{B}_5)$-measurable.

(ii) Suppose $X_1, \dots, X_5$ are independent. Show that T and S are independent.

[3] Suppose $X_1, X_2, \dots$ are independent random variables with $\mathbb{P}X_i = 0$ and $\sigma_i^2 := \mathbb{P}X_i^2 < \infty$ for each i. You may solve this problem either under the assumption that $\sup_i \sigma_i^2 \le K < \infty$ or under the weaker assumption $\sum_{i \in \mathbb{N}} \sigma_i^2/i^2 < \infty$. (The second case is a bit harder.) Define $S_n := X_1 + \dots + X_n$. Define $n(k) = 2^k$ and $B_k = \{n \in \mathbb{N} : n(k-1) < n \le n(k)\}$ for $k \in \mathbb{N}$.

Your aim is to prove that $S_n/n \to 0$ almost surely, as $n \to \infty$, by the following steps.

(i) Show that $\sum_{i \in \mathcal{N}} \mathbb{P}\{|X_i| > i\} < \infty$. Hint: Second moment bound.

(ii) Deduce that there exists a $\mathbb{P}$-negligible set $\mathcal{N}_0$ for which: to each $\omega \in \mathcal{N}_0^c$ there exists an $i_0(\omega) \in \mathbb{N}$ such that $|X_i(\omega)| \le i$ when $i \ge i_0(\omega)$.

(iii) Define $T_n = \sum_{I \le n} X_i\{|X_i| \le i\}$. Show that $(S_n - T_n)/n \to 0$ almost surely, and hence it suffices to prove $T_n/n \to 0$ almost surely.

(iv) Show that

$$|\mathbb{P}T_n/n| \le n^{-1} \sum\nolimits_{i \le n} \sigma_i^2/i \le \sum\nolimits_{i \le n} (\sigma_i^2/i^2) \min(1, i/n) \to 0.$$

Deduce that it suffices to prove $(T_n - \mathbb{P}T_n)/n \to 0$ almost surely.

(v) Define $Y_i := X_i^+\{|X_i| \le i\}$ and $Z_i := X_i^-\{|X_i| \le i\}$ and $\mu_i := \mathbb{P}Y_i$ and $\nu_i := \mathbb{P}Z_i$. Explain why it suffices to prove $\sum_{i \le n}(Y_i - \mu_i)/n \to 0$ almost surely and $\sum_{i \le n}(Z_i - \nu_i)/n \to 0$ almost surely.

(vi) Define $W_n := \sum_{i\le n} Y_i$ and $M_n := \mathbb{P}W_n$. Explain why

$$W_{n(k-1)} - M_{n(k)} \le W_n - M_n \le W_{n(k)} - M_{n(k-1)} \qquad \text{for } n \in B_k.$$

(vii) Deduce that

$$\max_{n\in B_k} \frac{|W_n - M_n|}{n} \le 2\frac{|W_{n(k)} - M_{n(k)}|}{n(k)} + |W_{n(k-1)} - M_{n(k-1)}| + \frac{M_{n(k)} - M_{n(k-1)}}{n}.$$

(viii) Use the second moment analog of the proof (given in class) for SLLN4 to show that $(W_{n(k)} - M_{n(k)})/n(k) \to 0$ almost surely as $k \to \infty$.

(ix) Show that

$$\frac{M_{n(k)} - M_{n(k-1)}}{n(k)} \le \sum\nolimits_{i\in B_k} \sigma_i^2/i^2 \to 0 \qquad \text{as } k \to \infty.$$

(x) Similar arguments work with $\sum_{i\le n}(Z_i - \nu_i)/n$. Combine the results from all the previous parts to exlain why $S_n/n \to 0$ almost surely.