Statistics 330b/600b, Math 330b spring 2016
Homework # 7
Due: Thursday 10 March

Throughout this sheet μ and ν are sigma-finite measures on $\mathcal{B}(\mathbb{R})$ and $\lambda = \mu \otimes \nu$. Define $\mathbb{A}_\mu = \{x \in \mathbb{R} : \mu\{x\} > 0\}$ and $\mathbb{A}_\nu = \{y \in \mathbb{R} : \nu\{y\} > 0\}$. When solving a problem you may use the results from previous problems, even if you did not solve those problems.

[1] Regarding atoms:

(i) Explain why $\mathcal{B}(\mathbb{R}^2) = \mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R})$. Deduce that $D := \{(x, y) \in \mathbb{R}^2 : x = y\}$ (the 'diagonal') belongs to $\mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R})$.

(ii) Show that the set $\mathbb{A}_\mu$ of atoms is at worst countably infinite. Hint: How many points x with $\mu\{x\} \geq 1/n$ can there be inside a set B with $\mu B < \infty$?

(iii) Explain why the function $h(x) := \nu^y\{x = y\}$ is well defined and $\mathcal{B}(\mathcal{R})$-measurable. Show that $h(x) = \sum_{\alpha \in \mathbb{A}_\mu} \nu\{\alpha\}\{x = \alpha\}$.

(iv) Show that $\lambda(D) = \sum_{\alpha \in \mathbb{A}_\mu \cap \mathbb{A}_\nu} \mu\{\alpha\}\nu\{\alpha\}$.

*[2] Suppose $f \in \mathcal{L}^1(\mathbb{R}, \mathcal{B}(\mathbb{R}), \mu)$ and $g \in \mathcal{L}^1(\mathbb{R}, \mathcal{B}(\mathbb{R}), \nu)$.

(i) Show that the function $\psi(x, y) = f(x)g(y)$ belongs to $\mathcal{L}^1(\mathbb{R}^2, \mathcal{B}(\mathbb{R}^2), \lambda)$.

(ii) Define $F(y) := \mu^x\left(f(x)\{x \leq y\}\right)$ and $G(x) := \nu^y\left(g(y)\{y \leq x\}\right)$. Show that

$$\mu^x\left(G(x)f(x)\right) + \nu^y\left(F(y)g(y)\right) = (\mu f)(\nu g) + \sum\nolimits_{\alpha \in \mathbb{A}_\mu \cap \mathbb{A}_\nu} f(\alpha)g(\alpha)\mu\{\alpha\}\nu\{\alpha\}.$$

(iii) (optional) Explain how the result from part (ii) corresponds to the classical formula for integration by parts. (You might need to read UGMTP Section 3.4 to come up with a convincing answer.)

*[3] Suppose T is a function from a set $\mathcal{X}$ into a set $\mathcal{Y}$, which is equipped with a σ-field $\mathcal{B}$. Recall that $\sigma(T) := \{T^{-1}B : B \in \mathcal{B}\}$ is the smallest sigma-field on $\mathcal{X}$ for which T is $\sigma(T)\backslash\mathcal{B}$-measurable.

Show that to each f in $\mathcal{M}^+(\mathcal{X}, \sigma(T))$ there exists a g in $\mathcal{M}^+(\mathcal{Y}, \mathcal{B})$ such that $f = g \circ T$ (that is, $f(x) = g(T(x))$, for all x in $\mathcal{X}$) by following these steps.

(i) If f is the indicator function of $T^{-1}(B)$ and g is the indicator function of B, show that $f = g \circ T$.

(ii) Extend to the case where $f \in \mathcal{M}^+_{\text{simple}}(\mathcal{X}, \sigma(T))$.

(iii) Suppose $f_n = g_n \circ T$ is a sequence in $\mathcal{M}^+_{\text{simple}}(\mathcal{X}, \sigma(T))$ that increases pointwise to f. Define $g(y) = \limsup g_n(y)$ for each y in $\mathcal{Y}$. Show that g has the desired property.

(iv) In part (iii), why can't you assume that $\lim g_n(y)$ exists for each y in $\mathcal{Y}$?

[4] Suppose X and Y are independent random variables for which $\mathbb{P}\{X = Y\} = 1$. Show that there exist a constant $a \in \mathbb{R}$ for which $\mathbb{P}\{X = a\} = 1$. Hint: Write P for the distribution of X and Q for the distribution of Y. Use Tonelli for a particularly elegant proof.

[5] Show that $\{(x, y) \in \mathbb{R}^2 : f(x) > y\} \in \mathcal{B}(\mathbb{R}^2)$ if $f \in \mathcal{M}^+(\mathbb{R}, \mathcal{B}(\mathbb{R}))$.