Statistics 330b/600b, Math 330b spring 2016
Homework # 9
Due: Thursday 7 April

*[1] *For this Problem do not assume any facts concerning the Radon-Nikodym theorem.*

Suppose $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space with $\mathcal{F} = \sigma(\mathcal{E})$ with $\mathcal{E} = \{E_i : i \in \mathbb{N}\}$. Suppose also that μ is a measure on $\mathcal{F}$ with $\mu F \le \mathbb{P}F$ for all $F \in \mathcal{F}$. Define $\mathcal{F}_n = \sigma\{E_1, \ldots, E_n\}$.

(i) Do you believe that $\mathcal{F}_n$ is also generated by a set π_n of at most 2^n "atoms", sets of the form $\cap_{i \le n} B_i$ with B_i equal to either E_i or E_i^c for each i?

(ii) Define

$$X_n(\omega) := \sum_{A \in \pi_n} \{\omega \in A, \mathbb{P}A > 0\} \frac{\mu A}{\mathbb{P}A}.$$

Show that $\{(X_n, \mathcal{F}_n) : n \in \mathbb{N}\}$ is a martingale with $0 \le X_n \le 1$ and $\mathbb{P}(X_n F) = \mu F$ for each $F \in \mathcal{F}_n$.

(iii) By the convergence theorem for positive (super)-martingales, there is an $\mathcal{F}$-measurable random variable X_∞ for which $X_n \to X_\infty$ almost surely. Prove that $\mathbb{P}(X_\infty F) = \mu F$ for each $F \in \mathcal{F}$. Hint: Generating class argument with $\mathcal{D} = \cup_{i \in \mathbb{N}} \mathcal{F}_i$.

*[2] Suppose $\{(X_i, \mathcal{F}_i) : i = 0, 1, \ldots, n\}$ is a martingale with $\mathbb{P}M_i^2 < \infty$ for each i. Define $M := \max_{i \le n} |X_i|$.

(i) For each $t > 0$ define $\sigma(t) := n \wedge \inf\{i : |X_i \ge t\}$. Show that $\sigma(t)$ is a stopping time for which

$$t\{M \ge t\} = t\{|X_{\sigma(t)}| \ge t\} \le |X_{\sigma(t)}\{|X_{\sigma(t)}| \ge t\}$$

(ii) Use the stronger form of the Stopping Time Lemma to deduce that

$$2t\mathbb{P}\{M \ge t\} \le \mathbb{P}|X_n|\{M \ge t\} \qquad \text{for each } t > 0.$$

(iii) Integrate the last inequality with respect to t then invoke the Hölder inequality to conclude that $\mathbb{P}M^2 \le 4\mathbb{P}X_n^2$. Hint: If you plan on dividing by $\sqrt{\mathbb{P}M^2}$ you should explain why this quantity is neither zero nor infinite.

[3] Let $\{\xi_i : i \in \mathbb{N}\}$ be a sequence of independent, identically distributed, integrable random variables defined on $(\Omega, \mathcal{F}, \mathbb{P})$. Define $S_n = \sum_{i \le n} \xi_i$ and

$$\mathcal{T}_n = \sigma\{S_n, \xi_{n+1}, \xi_{n+2}, \ldots\}.$$

That is, $\mathcal{T}_n$ is the smallest sigma-field on Ω for which S_n and each ξ_i for $i > n$ is $\mathcal{T}_n$-measurable.

(i) Define $\mathbb{G}_n$ to be the set of all random variables of the form $f_n(S_n) \times \prod_{i=1}^m f_{n+i}(\xi_{n+i})$ with m finite and each f_j a bounded $\mathcal{B}(\mathbb{R})$-measurable function. Explain why $\sigma(\mathbb{G}_n) = \mathcal{T}_n$.

(ii) For each bounded, $\mathcal{T}_n$-measurable random variable H prove that $\mathbb{P}\xi_i H = \mathbb{P}\xi_1 H$ for $i \le n$. Hint: functional π-λ.

(iii) Show that $\mathbb{P}(\xi_1 \mid \mathcal{T}_n) = S_n/n$ almost surely.

(iv) Deduce that $\{(S_i/i, \mathcal{T}_i) : i = -n, -n+1, \ldots, 1\}$ is a martingale for each n.

(v) Read (and understand) UGMPT Theorem 6.41. Explain why S_n/n converges almost surely (and in $\mathcal{L}^1$-norm) to a random variable W that is measurable with respect to the sigma-field $\mathcal{T}_\infty = \cap_{n \in \mathbb{N}} \mathcal{T}_n$.

(vi) Read (and understand) UGMPT Theorem 6.51 then explain why every B in $\mathcal{T}_\infty$ has either $\mathbb{P}B = 0$ or $\mathbb{P}B = 1$.

(vii) Explain why $S_n/n \to \mathbb{P}\xi_1$ almost surely.