0.1 Remark concerning Problem 11.1, spring 2016

It would have helped if I had already proved the following result.

<1> **Theorem.** *Suppose $W \in \mathcal{L}^1(\Omega, \mathcal{F}, \mathbb{P})$ and $\{\mathcal{F}_n : n \in \mathbb{N}\}$ is a filtration on the space. Define $W_n = \mathbb{P}_{\mathcal{F}_n} W$ and $\mathcal{F}_\infty = \sigma\left(\cup_{i\in\mathbb{N}}\mathcal{F}_i\right)$. Then $\{(W_n, \mathcal{F}_n) : n \in \mathbb{N}\}$ is a martingale that converges almost surely and in $\mathcal{L}^1$ to $W_\infty := \mathbb{P}_{\mathcal{F}_\infty} W$.*

> **Remark.** I omit most of the almost sure qualifications that, strictly speaking, are needed when working with $\mathbb{P}_{\mathcal{F}_i}$.

Proof Without loss of generality we may assume $W \geq 0$. (Equivalently, we could prove the result for W^+ and W^- then combine the two conclusions.) We may also assume that W is $\mathcal{F}_\infty$-measurable, because

$$W_n = \mathbb{P}_{\mathcal{F}_n} W = \mathbb{P}_{\mathcal{F}_n}\mathbb{P}_{\mathcal{F}_\infty} W = \mathbb{P}_{\mathcal{F}_n} W_\infty.$$

The equality $\mathbb{P}_{\mathcal{F}_i}(\mathbb{P}_{\mathcal{F}_j} W) = \mathbb{P}_{\mathcal{F}_i} W$ for $i < j$ establishes the martingale property. The nonnegativity assumption makes $\{W_n\}$ a positive martingale. By UGMTP Theorem 6.22, W_n converges (almost surely) to some nonnegative random variable Z in $\mathcal{L}^1(\Omega, \mathcal{F}_\infty, \mathbb{P})$. And by Corollary 6.24, the convergence also holds in $\mathcal{L}^1$ if $\mathbb{P}W_n \to \mathbb{P}Z$.

By Fatou's Lemma (and the fact that $\mathbb{P}W_i = \mathbb{P}W$ for all i) we already know that $\mathbb{P}Z \leq \mathbb{P}W$. It is enough to show that $\mathbb{P}Z \geq \mathbb{P}W - \epsilon$ for each $\epsilon > 0$.

By Monotone Convergence there exists some positive constant C for which $\mathbb{P}(W \wedge C) > \mathbb{P}W - \epsilon$. The sequence $W_{n,C} := \mathbb{P}_{\mathcal{F}_n}(W \wedge C)$ is also a positive martingale, which converges almost surely to some nonnegative Z_C in $\mathcal{L}^1(\Omega, \mathcal{F}_\infty, \mathbb{P})$. By monotonicity of conditional expectations, $W_n \geq W_{n,C}$ for all n, which implies $Z \geq Z_C$ and $\mathbb{P}Z \geq \mathbb{P}Z_C$.

All the $W_{n,C}$'s and the limit Z_C take values in the bounded interval $[0, C]$. By Dominated Convergence,

$$\mathbb{P}(W \wedge C) = \mathbb{P}W_{n,C} \to \mathbb{P}Z_C.$$

The inequality $\mathbb{P}Z \geq \mathbb{P}W - \epsilon$ and the convergence $\mathbb{P}|W_n - Z| \to 0$ follow.

For each F in $\mathcal{F}_i$, the martingale property and the $\mathcal{L}^1$ convergence imply (for $i < n$)

$$\mathbb{P}\left(WF\right) = \mathbb{P}\left(W_i F\right) = \mathbb{P}\left(W_n F\right) \to \mathbb{P}\left(ZF\right).$$

A π–λ argument then shows that $Z = W$ almost surely. (Remember the assumption that W is $\mathcal{F}_\infty$-mesurable.)

□