Generating classes of functions

<1> **Definition.** *Let $\mathcal{H}$ be a set of bounded, real-valued functions on a set $\mathcal{X}$. Call $\mathcal{H}$ a* ***$\boldsymbol{\lambda}$-space*** *if:*

(i) *$\mathcal{H}$ is a vector space*

(ii) *each constant function belongs to $\mathcal{H}$;*

(iii) *if $\{h_n\}$ is an increasing sequence of functions in $\mathcal{H}$ whose pointwise limit h is bounded then $h \in \mathcal{H}$.*

Define $\sigma(\mathcal{H})$ to be the smallest sigma-field on $\mathcal{X}$ for which each h in $\mathcal{H}$ is $\sigma(\mathcal{H})\backslash\mathcal{B}(\mathbb{R})$-measurable. Define $\mathcal{E} = \{\, \{x : h(x) > c\} : h \in \mathcal{H} \text{ and } c \in \mathbb{R}\}$ and $\mathcal{A} = \mathcal{A}_{\mathcal{H}} := \{A \subseteq \mathcal{X} : \mathbb{1}_A \in \mathcal{H}\}$.

- $\star$ Show that $\sigma(\mathcal{H}) = \sigma(\mathcal{E})$ and that $\mathcal{A}$ is a λ-class of sets.
- $\star$ Show that $\mathcal{H}$ is stable under uniform limits.

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For the following assertions assume that $\mathcal{H}$ is π-stable, that is, if $h_1, h_2 \in \mathcal{H}$ then the function $x \mapsto h_1(x)h_2(x)$ is also in $\mathcal{H}$.

- ⋆ Show that $\mathcal{A}$ is π-stable and hence $\mathcal{A}$ is a sigma-field.
- ⋆ Show that if $h \in \mathcal{H}$ then $h^+ \in \mathcal{H}$. Use the Weierstrass theorem to write h^+ as a uniform limit of functions $p_n(h)$, with p_n a polynomial.
- ⋆ Show that $\mathcal{H}$ is stable under pairwise maxima and pairwise minima. Use $h_1 \vee h_2 = h_1 + (h_2 - h_1)^+$.
- ⋆ Show that $\mathcal{E} \subseteq \mathcal{A}$. Use $\{h > c\} = \lim_n (1 \wedge n(h - c)^+)$.
- ⋆ Deduce that $\sigma(\mathcal{H}) = \sigma(\mathcal{E}) = \mathcal{A}$.
- ⋆ Deduce that $\mathcal{H}$ consists of the set of all bounded, $\mathcal{A}\backslash\mathcal{B}(\mathbb{R})$ measurable real functions on $\mathcal{X}$.

Now suppose that $\mathcal{H}$ is a λ-space and $\mathcal{G}$ is a π-stable subset of $\mathcal{H}$. Let $\mathcal{H}_0$ denote the smallest λ-space for which $\mathcal{H}_0 \supseteq \mathcal{G}$.

⋆ Imitate the proof of the π–λ-theorem for sets to show that $\mathcal{H}_0$ is π-stable.

⋆ Deduce that every bounded, $\sigma(\mathcal{G})\mathcal{B}(\mathbb{R})$-measurable function belongs to $\mathcal{H}$.